International Institute for
Technology and Management

Tutoring Sheet # 2a

SOLUTIONS

Unit 04b : Statistics 2

1. The discrete random variable X has the probability distribution specified in the following table:

x	-1	0	1	2
P(X =x)	0.25	0.10	0.45	0.20

a. Find P(-1 < X < 1) = P(X=0) = 0.10
   $P(-1 \le X < 1) = P(X=-1) + P(X=0) = 0.25 + 0.10 = 0.35$

b. Find E(2X + 3) = 2E(X) + 3
   E(X) = (-1)(0.25) + (0)(0.10) + (1)(0.45) + 2(0.20) = 0.6
   E(2X+3) = 2(0.6) + 3 = 4.2
   Var(2X + 3) = 4 Var(X) since Var(aX) = a^2 Var(X) and Var(3)=0
   Var(X) = $E(X^2) – E^2(X)$
   $E(X^2) = \sum x^2 P(X=x) = (-1)^2(0.25) + 0 + (1)(0.45) + 4(0.20) = 1.5$
   $Var(X) = 1.5 – (0.6)^2 = 1.14$, Var(2X + 3)=4 (1.14)=4.56

2. A bag contains 3 red balls and 1 blue ball.A second bag contains 1 red ball and 1 blue ball.A ball is picked out of each bag and is then placed in the other bag.What is the expected number of red balls in the first bag? ?(Assume that the balls are taken from each bag simultaneously).
Let X be the r.v. “the final number of red balls in the first bag”
Then , X can take the values 2 , 3 or 4 only.
P(X=2) = P(R1B2) = (3/4)(1/2) = 3/8
P(X=3) = P(R1R2) + P(B1B2) = (3/4)(1/2) + (1/4)(1/2) = ½
P(X=4) = P(B1R2) = (1/4)(1/2) = 1/8

x	2	3	4
P(X =x)	3/8	1/2	1/8

$E(X) = \sum x P(X=x) = 2(3/8) + 3(1/2) + 4(1/8) = 11/4$

3. For a discrete r.v. X , the c.d.f , F(X) is as shown:

x	1	2	3	4	5
$F(x)= P(X \le x)$	0.2	0.32	0.67	0.9	1

a. Find P(X = 3)

$F(3) = P(X \le 3) = P(X = 1)+P(X = 2)+P(X = 3) = 0.67$

$F(2) = P(X \le 2) = P(X = 1)+P(X = 2) = 0.32$

$P(X = 3) = F(3) – F(2) = 0.67 – 0.32 = 0.35$

b. $P(X > 2) = 1 – P(X \le 2) = 1 – F(2) = 1 – 0.32 = 0.68$

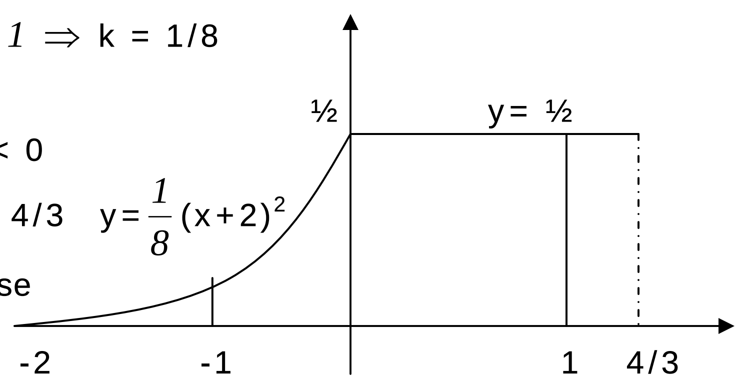
4. The continuous r.v. X has p.d.f. :

$$f(x)=\begin{cases} k(x+2)^2 & -2 \le x < 0 \\ 4k & 0 \le x \le 4/3 \\ 0 & \text{Otherwise} \end{cases}$$

a.) Find the value of the constant k and sketch the graph of y=f(x)

$$\int_{-2}^{0} k(x+2)^2 dx + \int_{0}^{4/3} 4k dx = 1 \Rightarrow k = 1/8$$

$$f(x)=\begin{cases} \frac{1}{8}(x+2)^2 & -2 \le x < 0 \\ 1/2 & 0 \le x \le 4/3 \\ 0 & \text{Otherwise} \end{cases}$$



b.) Find $P(-1 \le X \le 1) = P(-1 \le X \le 0) + P(0 \le X \le 1)$

$$P(-1 \le X \le 0) = \int_{-1}^{0} \frac{1}{8}(x+2)^2 dx = \frac{7}{24}$$

$P(0 \le X \le 1)$ = Area of rectangle = (1)(½) = ½

$$P(-1 \le X \le 1) = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$$

c.) Find P(X > 1)= Area of rectangle (x=1 to x= 4/3)=1/3 × ½= 1/6

5. The continuous r.v. X has p.d.f. , f(x) = 0.75 $(1+x^2)$, $0 \le x \le 1$
If E(X) = μ and Var(X) = σ^2, find P ($|X-\mu| < \sigma$)

E(X) = μ = $\int_0^1 0.75(1+x^2)dx = 0.5625$

$E(X^2)$ = $\int_0^1 0.75x^2(1+x^2)dx = 0.4$

Var(X) = σ^2 = $E(X^2) - E^2(X)$ = 0.0835 , $\sigma = \sqrt{Var(X)} = 0.289$

P ($|X-\mu| < \sigma$) = P($-\sigma < X - \mu < \sigma$) = P($\mu - \sigma < X < \mu + \sigma$)

= P (0.2735 < X < 0.8515) = $\int_{0.2735}^{0.8515} 0.75(1+x^2)dx = 0.583$

6. If X is a continuous r.v. with p.d.f , f(x) = $\frac{1}{8}x$, $0 \le x \le 4$

i.Find the c.d.f. F(x)

F(x) = $\int_0^x f(x)dx = \int_0^x \frac{1}{8}xdx = \frac{x^2}{16}$, $0 \le x \le 4$

ii.Find the median m

The median splits the area under the curve y = f(x) into two halves

i.e. $\int_0^m f(x)dx = ½$ or F(m) = ½ $\Rightarrow \frac{m^2}{16} = \frac{1}{2} \Rightarrow m^2 = 8$

$\Rightarrow$ m = $\pm$ 2.83 i.e. m = 2.83 since $0 \le m \le 4$

iii.Find P($0.3 \le X \le 1.8$) = F(1.8) – F(0.3) = 0.197

7. The c.d.f. F(x) for a continuous r.v. is defined as:

$$F(x) = \begin{cases} 0 & x < 0 \\ 2x - 2x^2 & 0 \le x \le ¼ \\ a + x & ¼ \le x \le ½ \\ b + 2x^2 - x & ½ \le x \le ¾ \\ 1 & x \ge ¾ \end{cases}$$

a. Find a and b

Since F(¾) = 1 for $x \ge$ ¾ ,

For ½ $\le x \le$ ¾ , $F(x) = b+2x^2 - x$ $\Rightarrow F(¾) = b + 2(¾)^2 - ¾ = 1$

$\Rightarrow b = 0.625$

For ½ $\le x \le$ ¾ , $F(0.5) = 0.625 + 2(0.625)^2 - 0.625 = 0.625$

For ¼ $\le x \le$ ½ , $F(x) = a + x$, $F(0.5) = a + 0.5 = 0.625$

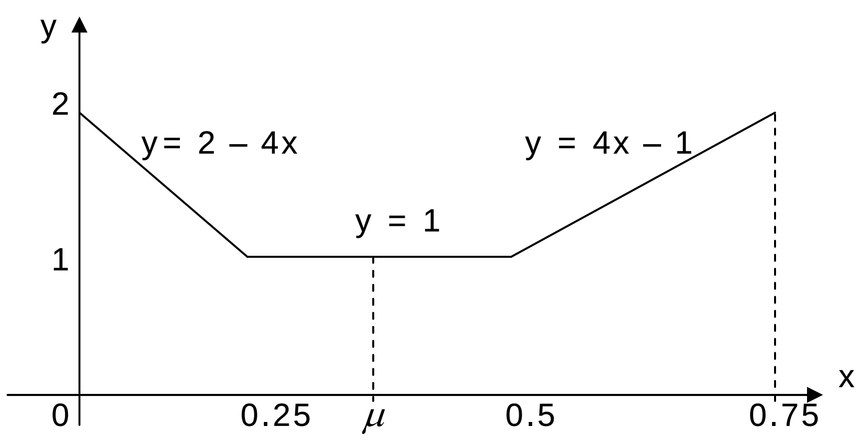
$\Rightarrow a = 0.125$

$$F(x)=\begin{cases} 0 & x<0 \\ 2x-2x^2 & 0 \le x \le \tfrac{1}{4} \\ 0.125+x & \tfrac{1}{4} \le x \le \tfrac{1}{2} \\ 0.625+2x^2-x & \tfrac{1}{2} \le x \le \tfrac{3}{4} \\ 1 & x \ge \tfrac{3}{4} \end{cases}$$

b. Find and sketch the p.d.f. f(x)

$$f(x)=\frac{dF(x)}{dx}$$

$$F(x)=\begin{cases} 0 & x<0 \\ 2-4x & 0 \le x \le \tfrac{1}{4} \\ 1 & \tfrac{1}{4} \le x \le \tfrac{1}{2} \\ 4x-1 & \tfrac{1}{2} \le x \le \tfrac{3}{4} \\ 1 & x > \tfrac{3}{4} \end{cases}$$



c. Find the mean μ

By symmetry : $\mu = 0.375$

8. The random variable X has density function f(x) given by

$$f(x) = 3e^{-3x}$$

defined over the region $x > 0$, Let $Y = e^X$, find E(Y) and Var(Y).

$$E(Y) = E(e^X) = \int_0^\infty e^x(3e^{-3x})dx = 3\int_0^\infty e^{-2x}dx = \frac{-3}{2}e^{-2x}\Big|_0^\infty = 0-(\frac{-3}{2}) = 1.5$$

$$E(Y^2) = E(e^{2X}) = \int_0^\infty e^{2x}(3e^{-3x})dx = 3\int_0^\infty e^{-x}dx = -3e^{-x}\Big|_0^\infty = 0-(-3) = 3$$

$$Var(Y) = E(Y^2) - E^2(Y) = 3 - (1.5)^2 = 0.75$$

9. A random variable can only take the values 0 and 2 , its variance is 1 , find its mean.

x	0	2
P(X =x)	p_0	p

$Var(X) = E(X^2) - \mu^2$, $E(X^2) = (0)^2(p_0) + (2)^2(p) = 4p$

$1 = 4p - \mu^2$ Now , $\mu = (0)(p_0) + 2p = 2p$

$1 = 4p - 4p^2 \Rightarrow p = ½ \Rightarrow \mu = 2p = 2(½) = 1$

10. X is a random variable with $E(X^2) = 3.6$ and $P(X = 2) = 0.6$ and $P(X = 3) = 0.1$. The random variable X takes just one other value besides 2 and 3.

i. What is the other value that X takes?

ii. What is the variance of X?

i.

x	2	3	c
P(X =x)	0.6	0.1	p

X is r.v. , $\sum P(X = x) = 1 \Rightarrow 0.6 + 0.1 + p = 1 \Rightarrow p = 0.3$

$E(X^2) = 4(0.6) + 9(0.1) + c^2(0.3) = 3.6 \Rightarrow c = 1$

ii. $Var(X) = E(X^2) - E^2(X)$

$E(X) = 2(0.6) + 3(0.1) + 1(0.3) = 1.8$

$Var(X) = E(X^2) - E^2(X) = 3.6 - (1.8)^2 = 0.36$

11. X is a random variable with density function

$$f_X(x) = 3(1+x)^2/19$$

over the range $(1,2)$. Using calculus, find the mean and the expected value of $1/(1+X)$. What is $P(X > 1.5)$?

$$E(\frac{1}{1+X}) = \int_1^2 \frac{1}{1+x} \times \frac{3(1+x)^2}{19} dx = \frac{3}{19}\int_1^2 (1+x)dx = \frac{15}{38}$$

$$P(X > 1.5) = \int_{1.5}^2 \frac{3(1+x)^2}{19} dx = \frac{3}{19}\int_1^2 (1+x)^2 dx = \frac{91}{152}$$

12. X is a random variable with density function

$$f_X(x) = a + bx$$

over the range $(0,1)$, where a, b are constants. The mean of X is 0.5.

i. Find a, b.
ii. Find the cumulative distribution function of X.
iii. Find the variance of X.

i. $\int_0^1 (a+bx)dx = 1 \Rightarrow$ a + ½ b = 1 , since μ = ½ :

$\int_0^1 x(a+bx)dx = 1/2 \Rightarrow$ ½ a + 1/3 b = ½

Solving the two equations : a = 1 , b = 0 , f(x) = 1

ii. $f(x) = \frac{dF(x)}{dx}$, F(x) = $\int_0^x f(x)dx = \int_0^x 1dx = x$ over (0,1)

iii. $E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$

$Var(X) = E(X^2) - E^2(X) = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$