International Institute for
Technology and Management

Tutoring Sheet # 1b

Unit 04b : Statistics 2

1. a) For each one of the statements below say whether the statement is true or false explaining your answer.

   i. For three events A, B and C, if A is independent of B and $C \subset A$, then C is independent of B.

   ii. If A and B are two events such that $\Pr(B|A) > \Pr(B)$, then $\Pr(A|B) > \Pr(A)$.

   iii. It is possible to find two independent events A and B such that $\Pr(B|A) > \Pr(B)$.

   vi. For two independent events A and B such that Pr(A) > 0 and Pr(B) > 0 , $\Pr(A \cup B) < \Pr(A) + \Pr(B)$

   v. If two events A and B are mutually exclusive then $\Pr(A^c \cap B^c) = 1 - \Pr(A) - \Pr(B)$

   vi. Two fair dice are thrown. The events $A = \{$the dice show a total for 2, 4 or 5$\}$ and $B = \{$the dice show a total score that is even$\}$ are independent.

i.False:toss two coins A: 1st coin showing a head , B: 2nd coin showing a head
C: both coins shows head

ii. True

$$P(B|A) > P(B) \Rightarrow \frac{P(B \cap A)}{P(A)} > P(B) \Rightarrow P(B \cap A) > P(A)P(B)$$

$$\Rightarrow \frac{P(B \cap A)}{P(B)} > P(A) \Rightarrow P(A|B) > P(A)$$

iii. False

If they are independent , $P(B \cap A) = P(A)P(B)$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)\,P(A)}{P(A)} = P(B)$$, which contradicts

The given : $P(B|A) > P(B)$.

iv. True

$P(A \cup B) = P(A) + P(B) - P(A \cap B) < P(A) + P(B)$

v. True

$$(A^c \cap B^c) = (A \cup B)^c \Rightarrow P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$$

Since A and B are mutually exclusive : $P(A \cup B) = P(A) + P(B)$

$$\Rightarrow P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - P(A) - P(B).$$

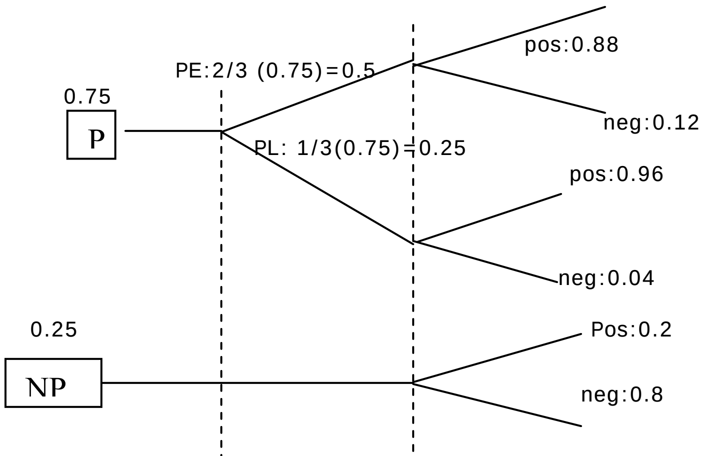
vi. True : $P(A) = \frac{1}{36} + \frac{3}{36} + \frac{4}{36} = \frac{8}{36} = \frac{2}{9}$, $P(B) = \frac{18}{36} = \frac{1}{2}$,

$P(A)\,P(B) = \frac{2}{9} \times \frac{1}{2} = \frac{1}{9}$, $A \cap B = \{2,4\}$

$$\Rightarrow P(A \cap B) = \frac{1}{36} + \frac{3}{36} = \frac{4}{36} = \frac{1}{9} \Rightarrow P(A \cap B) = P(A)P(B)$$

2. a) The proportion of pregnant women amongst women that take a particular pregnancy test is 75%. Two thirds of the pregnant women are at an early stage of pregnancy. If a woman that takes the test is pregnant at an early stage the test will be positive (indicate that she is pregnant) with probability 0.88. If she is pregnant at a later stage it will be positive with probability 0.96. If she is not pregnant it will be positive with probability 0.2.

   i. What is the probability the test will be positive?
   ii. Given that a woman took the test and it was positive, what is the probability that she is pregnant (at any stage)?
   iii. Given that a woman took the test and it was negative, what is the probability that she is pregnant (at any stage)?
   iv. What is the probability the test will show a false result?



Once one calculates the probabilities of simple events, then it becomes very clear. Let NP denote not pregnant, PE pregnant at an early stage, PL pregnant at a later stage, pos a positive result and neg a negative result. One quarter of women taking the test are not pregnant, half at an early stage and a quarter pregnant at a late stage. We have

$$\Pr(\text{NP} \,\&\, \text{pos}) = 0.25 \times 0.2 = 0.05$$
$$\Pr(\text{NP} \,\&\, \text{neg}) = 0.25 \times 0.8 = 0.2$$
$$\Pr(\text{PE} \,\&\, \text{pos}) = 0.5 \times 0.88 = 0.44$$
$$\Pr(\text{PE} \,\&\, \text{neg}) = 0.5 \times 0.12 = 0.06$$
$$\Pr(\text{PL} \,\&\, \text{pos}) = 0.25 \times 0.96 = 0.24$$
$$\Pr(\text{PL} \,\&\, \text{neg}) = 0.25 \times 0.04 = 0.01$$

i. $0.05+0.44+0.24=0.73$

ii. $\frac{0.44+0.24}{0.73}=0.932$

iii. $\frac{0.06+0.01}{1-0.73}=0.259$

iv. $0.05+0.06+0.01=0.12$

3. Two fair dice are thrown. If their total is not 2 or 12 , they are thrown once more. What is the probability that the final outcome is even.

The probability that the score is 2 or 12 is $\frac{1}{36}+\frac{1}{36}=\frac{1}{18}$.

The probability that it is not is $\frac{17}{18}$

The required probability is therefore

$$\frac{1}{18}+\left(\frac{17}{18}\right)\times\left(\frac{1}{2}\right)=\frac{19}{36}.$$

4. On his way to work Mr D buys newspaper A with probability 0.4, newspaper B with probability 0.3, or no newspaper with probability 0.3. If he buys newspaper A he brings it home in the evening with probability 0.4 and if he buys newspaper B he brings it home with probability 0.6. His behaviour on a particular day is independent of his behaviour on any other day.

i. He did not bring a newspaper home yesterday. What is the probability that he did not buy one?
ii. Given that he brought a newspaper home for the last two days, what is the probability it was the same newspaper on both days?
iii. What is the expected value of the number of newspapers he is going to bring home during the next 10 days? What is its variance?

i) The probability he did not buy a newspaper is 0.3. The probability he does not bring home a newspaper is $0.4 \times (1-0.4) + 0.3 \times (1-0.6) + 0.3 \times 1 = 0.66$. So the conditional probability he did not buy a newspaper given he did not bring one home is $\frac{0.3}{0.66} = \frac{5}{11}$.

ii) The probability he brought home newspaper A two days running is $0.4 \times 0.4 \times 0.4 \times 0.4 = 0.0256$. The same probability for B is $0.3 \times 0.6 \times 0.3 \times 0.6 = 0.0324$. The probability he brought A the first day and B the second day is $0.4 \times 0.4 \times 0.6 \times 0.3 = 0.0288$. The probability he brought home B and then A is also 0.0288. So the required probability is

$$\frac{0.0256+0.0324}{0.0256+0.0324+2\times 0.0288}=0.5017.$$

iii) The probability he brings a newspaper home on any day is $1-0.66 = 0.34$. The number of newspapers he brings home in the next days is Binomial with parameters 10 and 0.34. So the expected value is $10 \times 0.34 = 3.4$ and the variance is $10 \times 0.34 \times 0.66 = 2.244$.

5. A football team has three goalkeepers A, B and C. When A plays in a match the team does not concede a goal with probability 0.5. The same probability for B is 0.4 and for C 0.3. If the team does not concede a goal, the same goalkeeper is used for the next match. If the team does concede a goal, the probability that A plays in the next match is 0.5, the probability that B plays is 0.3 and the probability that C plays is 0.2. During the last match the team conceded a goal.

i. What is the probability that A will play for the next two matches?
ii. Given the same goalkeeper played for the last two matches, what is the probability that no goal was conceded in the first of those matches?

i. The probability that A will play, not concede a goal and play again is

$$0.5 \times 0.5 \times 1 = 0.25$$

and the probability that he will play, concede a goal and play again is

$$0.5 \times 0.5 \times 0.5 = 0.125$$

so the answer is $0.25 + 0.125 = 0.375$. (**4 marks**).

ii. The probability that B will play, not concede a goal and play again is

$$0.3 \times 0.4 \times 1 = 0.12$$

and the probability that he will play, concede a goal and play again is

$$0.3 \times 0.6 \times 0.3 = 0.054.$$

The probability that C will play, not concede a goal and play again is

$$0.2 \times 0.3 \times 1 = 0.06$$

and the probability that he will play, concede a goal and play again is

$$0.2 \times 0.7 \times 0.2 = 0.028$$

and so the answer is

$$\frac{0.25+0.12+0.06}{0.25+0.125+0.12+0.054+0.06+0.028} = \frac{0.43}{0.637} = 0.675.$$

6. i. State and prove Bayes' theorem.

Baye's Theorem

Let S be a sample space. If A_1, A, A_3 ... A_n are mutually exclusive and exhaustive events such that $P(A_i) \neq 0$ for all i.

Then for any event A which is a subset of

$$S = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n \text{ and } P(A) > 0$$

We have,

$$P(A_i / A) = \frac{P(A_i).P(A / A_i)}{\sum_{i=1}^{n} P(A_i)\, P(A / A_i)} \text{ for all } i = 1,2,3,\ldots, n$$

Proof:

We have $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$ and $A_i \cap A_j = \phi$ for $i \neq j$

Since $A \subseteq S$

$\Rightarrow \quad A = A \cap S$

$= A \cap (A_1 \cup A_2 \cup A_3 \cup \ldots. \cup A_n)$

$= (A \cap A_1) \cup (A \cap A_2) \cup (A \cap A_3) \cup \ldots. \cup (A \cap A_n)$

$\therefore \quad P(A) = P(A \cap A_1) + P(A \cap A_2) + P(A \cap A_3) + \ldots. + P(A \cap A_n)$

$[\because \ (A \cap A_i) \cap (A \cap A_j) = \phi, \text{ for } i \neq j]$

$P(A) = P(A_1)\, P(A/A_1) + P(A_2) P(A/A_2) + \ldots. + P(A_n)\, P(A/A_n)$ ….(1)

Also,

$P(A \cap A_i) = P(A)\ P(A_i / A)$ ….(2)

$$P(A_i / A) = \frac{P(A \cap A_i)}{P(A)}$$

$$\Rightarrow \quad P(A_i / A) = \frac{P(A_i) P(A / A_i)}{\sum_{i=1}^{n} P(A_i) P(A / A_i)} \qquad \text{(From (1) and (2))}$$

ii. First year students in statistics take a test of mathematical aptitude. Students taking the test are either well-prepared, or less well-prepared. Of the well-prepared students 95% will pass the test, whereas of the less well-prepared only 10% will pass. The pass rate for the test is 75%.
What is the proportion of students who are well-prepared?
Given that a student has passed the test, what is the probability that the student is not well-prepared?

Using Bayes' theorem , if π is the proportion of well prepared :

$$0.95\pi + 0.10(1 - \pi) = 0.75 \Rightarrow \pi = \frac{13}{17}$$

7.

Two students, C and D consider applying for a bursary, that may or may not be awarded, depending on their exam results. C makes an application, and will get the bursary with probability 3/4, provided D does not apply. D will apply with probability p, and if D does apply, the probability that C will get the bursary is 1/3.

i. What is the probability (in terms of p) that C will get the bursary?
ii. If the probability that D does not apply given that C is awarded the bursary is 3/7, what is the probability p?

It is easy to use the formula for total probability to obtain

$$P(C) = 3/4 - 5p/12.$$

Then

$$3/7 = P(D^c|C) = P(D^c \cap C)/P(C) = [3/4 \times (1-p)]/[3/4 - 5p/12]$$

Solving for p gives $p = 3/4$.

8.

A box contains 5 green balls and 5 red balls. A ball is chosen at random by a student from the box. If it is a green ball then the student writes down the answer 'green'. If the ball chosen is red, then the student writes down at random one of either 'red' or 'green'. Suppose the student has written down 'green'. What is the probability that a red ball was chosen? Suppose the first ball chosen is replaced, and another student acts in the same way as the first student. What is the probability that at least **one student choose a red ball if both students write down 'green'.**

This uses Bayes' Theorem. The probability that the student writes down 'green' is

$$\frac{1}{2} \times \frac{5}{10} + \frac{5}{10} = \frac{3}{4}.$$

So, the chance that the ball is red given that the student writes down 'red' is

$$\frac{\frac{1}{2} \times \frac{5}{10}}{\frac{3}{4}} = \frac{1}{3}.$$

When there are 2 students, the probability that both write down 'green' is $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. So, the chance that at least one student chose red is

$$\frac{\frac{1}{4}^2 + 2\frac{1}{4}\frac{1}{2}}{\frac{9}{16}} = \frac{5}{9}.$$