Chapter 2 – Probability

Basic Axioms and properties

1. $0 \le \Pr(A) \le 1$ 2. $\Pr(S) = 1$ 3. $A \subseteq B \Rightarrow \Pr(A) \le \Pr(B)$ 4. $\Pr(\phi) = 0$
5. If A and B are disjoint i.e. $A \cap B \neq \phi \Rightarrow P(A \cup B) = P(A) + P(B)$.
6. For any two events A and B, $\Pr(A \cup B) = \Pr(A)+\Pr(B) - \Pr(A \cap B)$
7. For any three events A, B, and C,
$\Pr(A \cup B \cup C) = \Pr(A)+\Pr(B)+\Pr(C) -\Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) +\Pr(A \cap B \cap C)$
8. $\Pr(A^c) = 1 - \Pr(A)$
9. $P(B) = P(B \cap A) + P(B \cap A^c)$
10. If $\Pr(B) > 0$, the conditional prob of A given B is $\Pr(A|B) \equiv \Pr(A \cap B)/\Pr(B)$.
Remark: If A and B are disjoint, then $\Pr(A|B) = \Pr(A \cap B)/\Pr(B)= 0/\Pr(B)= 0$.
11. Two events A and B are **independent** means:
   - $P(A|B)=P(A)$ or $P(B|A)=P(B)$
   - $P(A \cap B) = P(B)P(A|B) = P(A)P(B)$. since $P(A|B)=P(A)$

12. $P(B) = P(B|A) \times P(A) + P(B|A^c) \times P(A^c)$

13. Bayes Theorem: If $A_1, A_2, \ldots, A_n$ form a partition of S and B is any event, then

$$\Pr(A_i|B) = \Pr(A_i \cap B)/ \Pr(B) = \frac{\Pr(A_i)\Pr(B|A_i)}{\sum_{j=1}^{n}\Pr(A_j)\Pr(B|A_j)}$$

The $\Pr(A_i)$'s are prior probabilities ("before B").
The $\Pr(A_i|B)$'s are posterior probabilities ("after B").
Example1: In a certain city with good police,
Pr(Any defendent brought to trial is guilty) = 0.99. In any trial,
Pr(Jury acquits if defendent is innocent) = 0.95, Pr(Jury convicts if defendent is guilty) = 0.95.
Find Pr(Defendent is innocent|Jury acquits).
Events: I = "innocent", G = "guilty" = I^c, A = "acquittal".
Since the partition is {I,G}, Bayes' $\Rightarrow \Pr(I|A) = \Pr(I)\Pr(A|I)/ \Pr(I)\Pr(A|I)+\Pr(G)\Pr(A|G)$
$=(0.01)(0.95) / (0.01)(0.95)+(0.99)(0.05) = 0.161$.
Notice how the posterior prob's depend strongly on the prior prob's.
Example2: A store gets ½ of its items from Factory1, 1/4 from Factory 2, and 1/4 from Factory 3. 2% of Factory 1's items are defective. 2% of Factory 2's items are defective. 4% of Factory 3's items are defective. An item from the store is found to be bad. Find the prob it comes from Factory 1. [Answer should be< 1/2 since bad items favor Factory 3.]

Events: Fi = "Factory i", D = "defective item". Par-tition is {F1, F2, F3}.
$\Pr(F1|D) = \Pr(F1)\Pr(D|F1)/ \text{Sum}(j=1 \text{ to } 3)[\Pr(Fj)\Pr(D|Fj)]$
$=(0.5)(0.02)/ (0.5)(0.02)+(0.25)(0.02)+(0.25)(0.04) = 0.4$.
It turns out that $\Pr(F2|D) = 0.2$ and $\Pr(F3|D) = 0.4$.

Proofs

3. $A \subseteq B \Rightarrow \Pr(A) \leq \Pr(B)$

$\Pr(B) = \Pr(A \cup (A^c \cap B)) = \Pr(A) + \Pr(A^c \cap B) \geq \Pr(A)$.

4. $\Pr(\phi) = 0$

Since $A \cap \varnothing = \varnothing$, we have that A and $\varnothing$ are disjoint.

$\Pr(A) = \Pr(A \cup \varnothing) = \Pr(A) + \Pr(\varnothing) \Rightarrow \Pr(\varnothing) = \Pr(A) - \Pr(A) = 0$

6. For any two events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

First observe that $B = (A \cap B) \cup (A^c \cap B)$ where $A \cap B$ and $A^c \cap B$ are disjoint.

Thus $\Pr(B) = \Pr(A \cap B) + \Pr(A^c \cap B)$ (*)and so

$\Pr(A \cup B) = \Pr(A) + \Pr(A^c \cap B)$ (A, $A^c \cap B$ are disjoint)

$= \Pr(A) + \Pr(B) - \Pr(A \cap B)$ (by (*)).

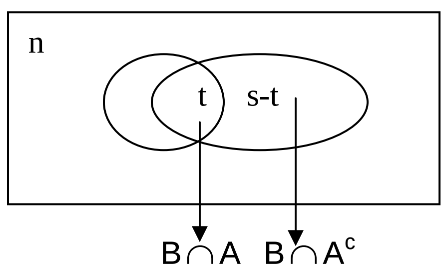
8. $\Pr(A^c) = 1 - \Pr(A)$

$1 = \Pr(S) = \Pr(A \cup A^c) = \Pr(A) + \Pr(A^c)$ since $A \cap A^c = \varnothing$

9. $P(B) = P(B \cap A) + P(B \cap A^c)$

Let $n(S) = n$, $n(B) = s$, $n(A \cap B) = t \Rightarrow n(B \cap A^c) = s - t$

$$P(B) = \frac{s}{n} = \frac{t + (s-t)}{n} = \frac{t}{n} + \frac{s-t}{n}$$
$$= P(B \cap A) + P(B \cap A^c)$$



12. $P(B) = P(B|A) \times P(A) + P(B|A^c) \times P(A^c)$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B|A) \times P(A)$$

$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)} \Rightarrow P(B \cap A^c) = P(B|A^c) \times P(A^c)$$

$$P(B) = P(B \cap A) + P(B \cap A^c)$$
$$= P(B|A) \times P(A) + P(B|A^c) \times P(A^c)$$

Questions and Answers

1. Define mutually exclusive events , independent events and explain the difference between them.

Two events, A and B, are **independent** if the fact that they can occur together and A occurs does not affect the probability of B occurring and vice versa.That is $A \cap B \neq \phi$

i.e. $P(A \cap B) \neq 0$ and $P(A \cap B) = P(A)P(B)$.

e.g. A: It rains on Mars tomorrow, B: Coin lands on H.

Replacing balls when drawn from a bag implies independent events.

Two events, A and B, are **mutually exclusive** if they can not occur together ,in this case $A \cap B = \phi$ i.e. $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$.

e.g. A and Ac mutually exclusive and exhaustive $Pr(A \cup A^c) = 1$

2. True or False : If $Pr(A) = 0$ then $A = \varnothing$.

We showed axiom 4 : $Pr(\phi) = 0$

Since $A \cap \varnothing = \varnothing$, we have that A and $\varnothing$ are disjoint.

$Pr(A) = Pr(A \cup \varnothing) = Pr(A)+Pr(\varnothing) \Longrightarrow Pr(\varnothing) = Pr(A) - Pr(A) = 0$

Converse is **false**: $Pr(A) = 0$ does not imply $A = \varnothing$.

Example: Pick a random number between 0 and 1.

3.Two non-zero probability events **can not be both** mutually exclusive and independent. **True**

If A and B are non-zero probability events , $P(A) \neq 0$, $P(B) \neq 0$.

A and B are mutually exclusive $\Longrightarrow$ $Pr(A|B) = Pr(A \cap B)/Pr(B)= 0/Pr(B)= 0 \neq P(A)$

Therefore, for two non-zero probability events, if they are mutually exclusive, then they are not independent.

A and B are independent $\Longrightarrow$ $Pr(A|B) = P(A) \Longrightarrow P(A \cap B) = P(B)P(A|B) = P(A)P(B) \neq 0$

Therefore, for two non-zero probability events, if they are independent, then they are not mutually exclusive.

Remark: A and B are both mutually exclusive and independent if and only if : $P(A)= 0$ or $P(B)= 0$ or both.

4. If A and B are mutually exclusive, then $P(A) + P(B) \leq 1$ .True

If A and B are mutually exclusive, $A \cap B = \phi$ i.e. $P(A \cap B) = 0$

Hence $P(A) + P(B) = P(A \cup B) \leq 1$

5. If $P(A)+P(B)>1$, then they can not be mutually exclusive.

$P(A)+P(B)>1$, we have $Pr(A \cup B) = Pr(A)+Pr(B) - Pr(A \cap B)$

$\Longrightarrow$ $Pr(A)+Pr(B) = Pr(A \cup B) + Pr(A \cap B) >1$ if $Pr(A \cap B) = 0$ then $Pr(A \cup B) >1$ which is impossible , therefore $Pr(A \cap B) \neq 0$ and they can not be mutually exclusive.

6. For any two events A and B :

(a) $Pr(A\cup B) \leq P(A) + P(B)$

(b) $Pr(A \cap B) \geq P(A) + P(B) - 1$

(a) we have $Pr(A \cup B) = Pr(A)+Pr(B) - Pr(A \cap B) \leq P(A) + P(B)$ since $Pr(A \cap B) \geq 0$

(b) Since $Pr(A \cup B) \leq 1 \Rightarrow Pr(A)+Pr(B) - Pr(A \cap B) \leq 1 \Rightarrow Pr(A)+Pr(B) - 1 \leq Pr(A \cap B)$

$\Rightarrow Pr(A \cap B) \geq P(A) + P(B) - 1$

Remark : General form of (a) : $Pr(A\cup B\cup C) \leq P(A) + P(B) + P(C)$

General form of (b): $Pr(A \cap B\cap C) \geq P(A) + P(B)+P(C) - 2$

For n events : $Pr(A_1 \cap A_2\cap\ldots\cap A_n) \geq P(A_1) + P(A_2)+\ldots+P(A_n) - (n-1)$

7. If $P(A)+P(B)>1$, then they can not be independent. False

Using 6(b) :

Since $Pr(A \cup B) \leq 1 \Rightarrow Pr(A)+Pr(B) - Pr(A \cap B) \leq 1 \Rightarrow Pr(A)+Pr(B) - 1 \leq Pr(A \cap B)$

$\Rightarrow Pr(A \cap B) \geq P(A) + P(B) - 1 \Rightarrow Pr(A \cap B) > 0$ since $P(A) + P(B) > 1$

Therefore $Pr(A \cap B) > 0$ and A , B are independent.

8. Use Bayes' Theorem to show that if P(A), P(B) > 0 and P(A) < P(A|B) then P(B) < P(B|A).
Start with P(A) < P(A|B), and so, multiply through by P(B) > 0, giving by Bayes' Theorem

$P(A)P(B) < P(A|B)P(B) = P(A \cap B)$

Since P(A) > 0, divide through by P(A), giving by Bayes' Theorem

$P(B) < P(A \cap B) / P(A) = P(B|A)$ as required.

9. $\Omega = \{\{a,b\}, \{a.c\},\{b.c\}\}$ is a sample space. True

A sample space could have sets as its elements. For example if we consider an experiment that takes samples of size 2 from a population with 3 numbers a , b and c , then the sample space must contain the sets {a,b}, {a.c},{b.c}.

10.

If $P(\overline{A} \cup \overline{B}) = \frac{5}{6}$, $P(A) = \frac{1}{2}$ and $P(\overline{B}) = \frac{2}{3}$, are the events A and B independent ?

$$P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$\therefore \quad P(A \cap B) = 1 - P(\overline{A} \cup \overline{B}) = 1 - \frac{5}{6} = \frac{1}{6}$$

Again, $P(B) = 1 - P(\overline{B}) = 1 - \frac{2}{3} = \frac{1}{3}$

$$\therefore \quad P(A)P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \cap B)$$

Hence, the events A and B are independent.

Exercises (Answers provided at the end)

1. If A and B are independent events then so are :
(a) A^c and B (b) A^c and B^c (c) A and B^c

2. Suppose that A, B, and C are three independent events such that Pr(A) = 1/4, Pr(B) = 1/3, and Pr(C) = 1/2.
(a) Find the probability that none of the events will occur.
(b) Find the probability that exactly one of the events will occur.

3. Box A has 4 blue and 6 red sox. Box B has 5 blues and 5 reds. Let's roll a die. If the die's outcome is odd, then a sock from A is selected; if the outcome is even, a sock from B is selected. Suppose that a blue sock is selected. What is the probability that the die toss was even?

4. Three boxes of the same appearance have the following proportions of black and white balls : Box I – 5 black and 3 white; Box II – 6 black and 2 white; Box III – 3 black and 5 white. One of the boxes is selected at random and one ball is drawn randomly from it. (*i*) What is the probability that the ball is black ? (*ii*) Given that the ball is black, find the probability that it came from Box III.

5. Suppose all possible four outcomes e_1 , e_2 , e_3 , e_4 of an experiment are equally likely. Define the events A , B, C as $A=\{e_1,e_4\}$, $B=\{e_2,e_4\}$, $C=\{e_3,e_4\}$ .What can you say about Dependence or independence of the events A,B and C .

6. Find the probability that in a throw of two dice ,the outcome is even or less than 5.

7. Two newspapers are published in a certain city , it is estimated by a survey that 16% read X , 14% read Y and 5% read both.Find the probability that a randomly selected person :
(i) doesn't read any newspaper (ii) reads only Y

8. There are five prisoners, two of whom are to receive a pardon and be released from prison. Two white balls and three black balls are placed in a bag with each prisoner drawing a ball from the bag. The two prisoners who pick white balls are released. Suppose that the five prisoners take it in turns to draw a ball from the bag:
(a) What is the probability that the first prisoner to draw a ball from the bag is released?
(b) What is the probability that the second prisoner to draw a ball from the bag is released given that the first prisoner to draws a white ball from the bag?
(c) What is the probability that the third prisoner to draw a ball from the bag is released given that the first prisoner to draws a white ball from the bag?
(d) What is the probability that the last prisoner to draw a ball from the bag is released?

9. Toss a fair coin until the first head appears. Find the probability that an odd number of tosses is required to achieve that.

Exercises Solutions

1. If A and B are independent events then so are :

(a) A^c and B (b) A^c and B^c (c) A and B^c

You'll be able to see these clearly with the aid of Venn diagrams.

(a) Since $A^c \cap B$ and $A \cap B$ are mutually exclusive ,
$P(B) = P(B \cap A) + P(B \cap A^c)$ (see axiom 9)
$P(B \cap A^c) = P(B) - P(B \cap A)$ but A and B are independent , $P(B \cap A)=P(A)P(B)$
$P(B \cap A^c) = P(B) - P(A)P(B) = P(B)(1-P(A))= P(B)P(A^c)$
Therefore A^c and B are independent.

(b) We use Demorgan's : $A^c \cap B^c = (A \cup B)^c$
$P(A^c \cap B^c)=P(A \cup B)^c = 1 - P(A \cup B)= 1- [P(A)+P(B)-P(A\cap B)]$
$P(A^c \cap B^c)= 1- P(A)- P(B)+ P(A\cap B)$ but $P(A\cap B)=P(A)P(B)$
$P(A^c \cap B^c)= 1- P(A)- P(B)+ P(A)P(B) = 1- P(A) - P(B)[1 - P(A)]$
$P(A^c \cap B^c)= [1- P(A)][1 - P(B)]= P(A^c)P(B^c)$ .Therefore A^c and B^c are independent

(c) $P(A) = P(A\cap B) + P(A\cap B^c)$
$P(A\cap B^c)= P(A) - P(A\cap B) = P(A) - P(A)P(B) = P(A)[1 - P(B)]$
$P(A\cap B^c)= P(A)P(B^c)$ .Therefore A and B^c are independent.

2. (a) $Pr(A^c \cap B^c \cap C^c) = Pr(A^c)Pr(B^c)Pr(C^c) = (3/4)(2/3)(1/2) = 1/4$
(b) $Pr(A^c \cap B^c \cap C) + Pr(A^c \cap B\cap C^c) + Pr(A \cap B^c \cap C^c)$
$= Pr(A^c)Pr(B^c)Pr(C) + Pr(A^c)Pr(B)Pr(C^c) + Pr(A)Pr(B^c)Pr(C^c)$
$= 11/ 24$

3. By Bayes' rule,
Pr(Even | Blue) = Pr(Even) Pr(Blue|Even)/Pr(Even)Pr(Blue|Even) + Pr(Odd) Pr(Blue|Odd)
= ½ . ½ / (½ . ½ + ½ . 2/5) = 5/9

4. Let A_1, A_2 and A_3 denote the selection of box I, box II and box III respectively, and B denote the event of drawing a black ball.

Here the events A_1, A_2 and A_3 are exhaustive and mutually exclusive. As the box is selected at random,

4. $$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}.$$

Also $P(B \mid A_1) = \frac{5}{8}$, $P(B \mid A_2) = \frac{6}{8}$ and $P(B \mid A_3) = \frac{3}{8}$.

(*i*) The required probability is

$$P(B) = \sum_{i=1}^{3} P(A_i).P(B|A_i) = \frac{1}{3}\times\frac{5}{8}+\frac{1}{3}\times\frac{6}{8}+\frac{1}{3}\times\frac{3}{8} = \frac{1}{3}\times\frac{14}{8}=\frac{7}{12}.$$

(*ii*) We require $P(A_3/B) = \dfrac{P(A_3).P(B/A_3)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$, by Bayes' theorem.

$$= \frac{\frac{1}{3}.\frac{3}{8}}{\frac{7}{12}} = \frac{3}{14}.$$

5. Here the sample space is $S = \{e_1, e_2, e_3, e_4\}$, and $A = \{e_1, e_4\}$, $B = \{e_2, e_4\}$, $C = \{e_3, e_4\}$.

$$\therefore \; P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}.$$

Again, $A \cap B = \{e_4\}$, $A \cap C = \{e_4\}$, and $B \cap C = \{e_4\}$.

$$\therefore \; P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}.$$

Thus, $P(A \cap B) = \frac{1}{4} = P(A).\,P(B)$,

$$P(A \cap C) = \frac{1}{4} = P(A)\,.\,P(C),$$

and $P(B \cap C) = \frac{1}{4} = P(B)\,.\,P(C)$.

So, the events A, B, C are pair-wise independent.

We also see that

$$A \cap B \cap C = \{e_4\}.$$

$$\therefore \; P(A \cap B \cap C) = \frac{1}{4} \neq P(A)\,.\,P(B)\,.\,P(C) = \frac{1}{8}.$$

Hence, the events A, B, C are not mutually independent, though they are pair-wise independent.

6. There are $6^2 = 36$ elementary events of the experiment, and these are equally likely as the dice are unbiased. Now, let E_1 and E_2 denote respectively the events of obtaining an even sum and a sum less than 5. Then, there are 18 elementary events that are favourable to E_1 and 6 [viz. (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2)] favourable to E_2, and 4 [viz. (1, 1), (1, 3), (3, 1), (2, 2)] favourable to $E_1 \cap E_2$.

$$\therefore \; P(E_1) = \frac{18}{36},\; P(E_2) = \frac{6}{36} \text{ and } P(E_1 \cap E_2) = \frac{4}{36}.$$

We require

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{18}{36} + \frac{6}{36} - \frac{4}{36} = \frac{20}{36} = \frac{5}{9}.$$

7. Let A_1 and A_2 denote that the selected person reads X and reads Y, respectively. Since the person is randomly chosen,

$$P(A_1) = 0.16,\ P(A_2) = 0.14,\ P(A_1 \cap A_2) = 0.05.$$

(*i*) The probability that the chosen person does not read any newspaper is

$$\begin{aligned} P(A_1^c \cap A_2^c) &= P(A_1 \cup A_2)^c \\ &= 1 - P(A_1 \cup A_2) \\ &= 1 - P(A_1) - P(A_2) + P(A_1 \cap A_2) \\ &= 1 - 0.16 - 0.14 + 0.05 \\ &= 0.75. \end{aligned}$$

(*ii*) In this case, the required probability is

$$\begin{aligned} P(A_1^c \cap A_2) &= P(A_2) - P(A_1 \cap A_2) \\ &= 0.14 - 0.05 \\ &= 0.09. \end{aligned}$$

8. (a) P(A) = 2 / 5 . Chance of choosing a white ball.

(b) P(B|A) = 1 / 4 .

Given that the first ball is white, it leaves four balls one of which is white.

(c) $P(C|A) = P(C|B \cap A)P(B|A) + P(C|B^c \cap A)P(B^c|A) = 0/3 \times 1/4 + 1/3 \times 3/4 = 1/4$

(d) P(E) = 2 / 5 .

The chances that the last ball is white is equal to the chance that the first ball is white.

9. Suppose the H appears in the first trial ,then $p_1 = ½$
If it appears in the second trial : T H , then $p_2 = (½)(½) = ¼$
If it appears in the third trial : T T H , then $p_3 = (½)(½)(½) = 1/8$
And so on …..
Let A be the event : an even number of tosses is required

$$P(A) = \sum_{i=even} \frac{1}{2^i} = \frac{1}{2^2} + \frac{1}{2^4} + \ldots\ldots.$$

Which is an infinite geometric progression of ratio r = ¼ and first term a = ¼

$$P(A) = \frac{a}{1-r} = \frac{1/4}{1-1/4} = \frac{1}{3}$$

Probability an odd number of tosses is required:

$P(A^c) = 1 – P(A) = 1- 1/3 = 2/3$