Chapter 2 – Probability

Basic Axioms and properties

1. $0 \leq \Pr(A) \leq 1$
2. $\Pr(S) = 1$
3. $A \subseteq B \Rightarrow \Pr(A) \leq \Pr(B)$
4. $\Pr(\emptyset) = 0$
5. If $A$ and $B$ are disjoint i.e. $A \cap B \neq \emptyset \Rightarrow \Pr(A \cup B) = \Pr(A) + \Pr(B)$
6. For any two events $A$ and $B$, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
7. For any three events $A$, $B$, and $C$,
   \[ \Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C) \]
8. $\Pr(A^c) = 1 - \Pr(A)$
9. $\Pr(B) = \Pr(B \cap A) + \Pr(B \cap A^c)$
10. If $\Pr(B) > 0$, the conditional prob of $A$ given $B$ is $\Pr(A|B) \equiv \Pr(A \cap B)/\Pr(B)$.

Remark: If $A$ and $B$ are disjoint, then $\Pr(A|B) = \Pr(A \cap B)/\Pr(B) = 0/\Pr(B) = 0$.

Two events $A$ and $B$ are independent means:
   • $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$
   • $\Pr(A \cap B) = \Pr(B)\Pr(A|B) = \Pr(A)\Pr(B)$ since $\Pr(A|B) = \Pr(A)$

12. $\Pr(B) = \Pr(B|A) \times \Pr(A) + \Pr(B|A^c) \times \Pr(A^c)$

13. Bayes Theorem: If $A_1, A_2, \ldots$, form a partition of $S$ and $B$ is any event, then

\[
\Pr(A_i|B) = \frac{\Pr(A_i)\Pr(B|A_i)}{\sum_{j=1}^{n} \Pr(A_j)\Pr(B|A_j)}
\]

The $\Pr(A_i)$’s are prior probabilities (“before B”).
The $\Pr(A_i|B)$’s are posterior probabilities (“after B”).

Example 1: In a certain city with good police,
$\Pr(\text{Any defendant brought to trial is guilty}) = 0.99$. In any trial,
$\Pr(\text{Jury acquits if defendant is innocent}) = 0.95$, $\Pr(\text{Jury convicts if defendant is guilty}) = 0.95$.
Find $\Pr(\text{Defendent is innocent | Jury acquits})$.

Events: $I = \text{“innocent”}$, $G = \text{“guilty”} = I^c$, $A = \text{“acquittal”}$.

Since the partition is \{I, G\}, Bayes’ $\Rightarrow \Pr(I|A) = \Pr(I)\Pr(A|I)/\Pr(I)\Pr(A|I) + \Pr(G)\Pr(A|G)$
\[= (0.01)(0.95) / (0.01)(0.95) + (0.99)(0.05) = 0.161. \]

Notice how the posterior prob’s depend strongly on the prior prob’s.

Example 2: A store gets $1/2$ of its items from Factory 1, $1/4$ from Factory 2, and $1/4$ from Factory 3.
2% of Factory 1’s items are defective. 2% of Factory 2’s items are defective. 4% of Factory 3’s items are defective.
An item from the store is found to be bad. Find the prob it comes from Factory 1.

[Answer should be < 1/2 since bad items favor Factory 3.]

Events: $F_i = \text{“Factory i”, D = “defective item”}$. Par-tition is \{F1, F2, F3\}.
$\Pr(F_i|D) = \Pr(F_i)\Pr(D|F_i) / \sum_{j=1}^{3} \Pr(F_j)\Pr(D|F_j)$
\[= (0.5)(0.02) / (0.5)(0.02) + (0.25)(0.02) + (0.25)(0.04) = 0.4. \]

It turns out that $\Pr(F2|D) = 0.2$ and $\Pr(F3|D) = 0.4$. 

1
Proofs

3. $A \subseteq B \Rightarrow Pr(A) \leq Pr(B)$
   
   $Pr(B) = Pr(A \cup (A^c \cap B)) = Pr(A) + Pr(A^c \cap B) \geq Pr(A)$.

4. $Pr(\emptyset) = 0$
   
   Since $A \cap \emptyset = \emptyset$, we have that $A$ and $\emptyset$ are disjoint.
   
   $Pr(A) = Pr(A \cup \emptyset) = Pr(A) + Pr(\emptyset) \Rightarrow Pr(\emptyset) = Pr(A) - Pr(A) = 0$

6. For any two events $A$ and $B$, $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

   First observe that $B = (A \cap B) \cup (A^c \cap B)$ where $A \cap B$ and $A^c \cap B$ are disjoint.

   Thus $Pr(B) = Pr(A \cap B) + Pr(A^c \cap B)$ (*) and so
   
   $Pr(A \cup B) = Pr(A) + Pr(A^c \cap B)$ ($A, A^c \cap B$ are disjoint)
   
   $= Pr(A) + Pr(B) - Pr(A \cap B)$ (by (*)).

8. $Pr(A^c) = 1 - Pr(A)$

   $1 = Pr(S) = Pr(A \cup A^c) = Pr(A) + Pr(A^c)$ since $A \cap A^c = \emptyset$

9. $P(B) = P(B \cap A) + P(B \cap A^c)$

   Let $n(S) = n$, $n(B) = s$, $n(A \cap B) = t \Rightarrow n(B \cap A^c) = s - t$

   $$P(B) = \frac{s}{n} = \frac{t + (s - t)}{n} = \frac{t}{n} + \frac{s - t}{n}$$
   
   $= P(B \cap A) + P(B \cap A^c)$

12. $P(B) = P(B \mid A) \times P(A) + P(B \mid A^c) \times P(A^c)$

   $$P(B \mid A) = P\left(\frac{B \cap A}{A}\right) \Rightarrow P(B \cap A) = P(B \mid A) \times P(A)$$

   $$P(B \mid A^c) = P\left(\frac{B \cap A^c}{A^c}\right) \Rightarrow P(B \cap A^c) = P(B \mid A^c) \times P(A^c)$$

   $P(B) = P(B \cap A) + P(B \cap A^c)$
   
   $= P(B \mid A) \times P(A) + P(B \mid A^c) \times P(A^c)$
Questions and Answers

1. Define mutually exclusive events, independent events and explain the difference between them.

Two events, A and B, are independent if the fact that they can occur together and A occurs does not affect the probability of B occurring and vice versa. That is \( A \cap B \neq \emptyset \)

i.e. \( P(A \cap B) \neq 0 \) and \( P(A \cap B) = P(A)P(B) \).

e.g. A: It rains on Mars tomorrow, B: Coin lands on H.

Replacing balls when drawn from a bag implies independent events.

Two events, A and B, are mutually exclusive if they can not occur together, in this case \( A \cap B = \emptyset \) i.e. \( P(A \cap B) = 0 \) and \( P(A \cup B) = P(A) + P(B) \).

e.g. A and Ac mutually exclusive and exhaustive \( P(A \cup A^c) = 1 \)

2. True or False: If \( Pr(A) = 0 \) then \( A = \emptyset \).

We showed axiom 4: \( Pr(\emptyset) = 0 \)

Since \( A \cap \emptyset = \emptyset \), we have that A and \( \emptyset \) are disjoint.

\( Pr(A) = Pr(A \cup \emptyset) = Pr(A) + Pr(\emptyset) \Rightarrow Pr(\emptyset) = Pr(A) - Pr(A) = 0 \)

Converse is false: \( Pr(A) = 0 \) does not imply \( A = \emptyset \).

Example: Pick a random number between 0 and 1.

3. Two non-zero probability events can not be both mutually exclusive and independent. True

If A and B are non-zero probability events, \( P(A) \neq 0 \), \( P(B) \neq 0 \).

A and B are mutually exclusive \( \Rightarrow \) \( Pr(A | B) = Pr(A \cap B)/Pr(B) = 0/Pr(B) = 0 \neq P(A) \)

Therefore, for two non-zero probability events, if they are mutually exclusive, then they are not independent.

A and B are independent \( \Rightarrow \) \( Pr(A | B) = P(A) \Rightarrow P(A \cap B) = P(B)P(A | B) = P(A)P(B) \neq 0 \)

Therefore, for two non-zero probability events, if they are independent, then they are not mutually exclusive.

Remark: A and B are both mutually exclusive and independent if and only if: \( P(A) = 0 \) or \( P(B) = 0 \) or both.

4. If A and B are mutually exclusive, then \( P(A) + P(B) \leq 1 \). True

If A and B are mutually exclusive, \( A \cap B = \emptyset \) i.e. \( P(A \cap B) = 0 \)

Hence \( P(A) + P(B) = P(A \cup B) \leq 1 \)

5. If \( P(A) + P(B) > 1 \), then they can not be mutually exclusive.

\( P(A) + P(B) > 1 \), we have \( Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) \)

\( \Rightarrow Pr(A) + Pr(B) = Pr(A \cup B) + Pr(A \cap B) > 1 \) if \( Pr(A \cap B) = 0 \) then \( Pr(A \cup B) > 1 \) which is impossible, therefore \( Pr(A \cap B) \neq 0 \) and they can not be mutually exclusive.
6. For any two events A and B :
(a) \( \Pr(A \cup B) \leq \Pr(A) + \Pr(B) \)
(b) \( \Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1 \)

(a) we have \( \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \leq \Pr(A) + \Pr(B) \) since \( \Pr(A \cap B) \geq 0 \)
(b) Since \( \Pr(A \cup B) \leq 1 \Rightarrow \Pr(A) + \Pr(B) - \Pr(A \cap B) \leq 1 \Rightarrow \Pr(A) + \Pr(B) - 1 \leq \Pr(A \cap B) \)
\( \Rightarrow \Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1 \)

**Remark:** General form of (a) : \( \Pr(A \cup B \cup C) \leq \Pr(A) + \Pr(B) + \Pr(C) \)

General form of (b): \( \Pr(A \cap B \cap C) \geq \Pr(A) + \Pr(B) + \Pr(C) - 2 \)

For n events : \( \Pr(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \Pr(A_1) + \Pr(A_2) + \ldots + \Pr(A_n) - (n-1) \)

7. If \( \Pr(A) + \Pr(B) > 1 \), then they can not be independent. **False**

Using (b) :
Since \( \Pr(A \cup B) \leq 1 \Rightarrow \Pr(A) + \Pr(B) - \Pr(A \cap B) \leq 1 \Rightarrow \Pr(A) + \Pr(B) - 1 \leq \Pr(A \cap B) \)
\( \Rightarrow \Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1 \) since \( \Pr(A) + \Pr(B) > 1 \)

Therefore \( \Pr(A \cap B) > 0 \) and A , B are independent.

8. Use Bayes’ Theorem to show that if \( \Pr(A) > 0 \), \( \Pr(B) > 0 \) and \( \Pr(A) < \Pr(A | B) \) then \( \Pr(B) < \Pr(B | A) \).

Start with \( \Pr(A) < \Pr(A | B) \), and so, multiply through by \( \Pr(B) > 0 \), giving by Bayes’ Theorem
\( \Pr(A) \Pr(B) < \Pr(A | B) \Pr(B) = \Pr(A \cap B) \)

Since \( \Pr(A) > 0 \), divide through by \( \Pr(A) \), giving by Bayes’ Theorem
\( \Pr(B) < \Pr(A \cap B) / \Pr(A) = \Pr(B | A) \) as required.

9. \( \Omega = \{ \{a, b\}, \{a, c\}, \{b, c\} \} \) is a sample space. **True**

A sample space could have sets as its elements. For example if we consider an experiment that takes samples of size 2 from a population with 3 numbers a , b and c , then the sample space must contain the sets \( \{a, b\}, \{a, c\}, \{b, c\} \).

10. If \( \Pr(\overline{A} \cup \overline{B}) = \frac{5}{6} \), \( \Pr(A) = \frac{1}{2} \) and \( \Pr(B) = \frac{2}{3} \), are the events A and B independent ?

    \( \Pr(\overline{A} \cup \overline{B}) = \Pr(\overline{A} \cap \overline{B}) = 1 - \Pr(A \cap B) \)

\[ \therefore \Pr(A \cap B) = 1 - \Pr(\overline{A} \cup \overline{B}) = 1 - \frac{5}{6} = \frac{1}{6} \]

Again, \( \Pr(B) = 1 - \Pr(\overline{B}) = 1 - \frac{2}{3} = \frac{1}{3} \)

\[ \therefore \Pr(A) \Pr(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = \Pr(A \cap B) \]

Hence, the events A and B are independent.
Exercises (Answers provided at the end)

1. If A and B are independent events then so are:
   (a) $A^c$ and $B$
   (b) $A^c$ and $B^c$
   (c) $A$ and $B^c$

2. Suppose that A, B, and C are three independent events such that $Pr(A) = 1/4$,
   $Pr(B) = 1/3$, and $Pr(C) = 1/2$.
   (a) Find the probability that none of the events will occur.
   (b) Find the probability that exactly one of the events will occur.

3. Box A has 4 blue and 6 red sox. Box B has 5 blues and 5 reds. Let’s roll a die. If the
die’s outcome is odd, then a sock from A is selected; if the outcome is even, a sock
from B is selected. Suppose that a blue sock is selected. What is the probability
that the die toss was even?

4. Three boxes of the same appearance have the following proportions of black and white balls:
   Box I – 5 black and 3 white; Box II – 6 black and 2 white; Box III – 3 black and 5 white. One of
   the boxes is selected at random and one ball is drawn randomly from it. (i) What is the probability
   that the ball is black? (ii) Given that the ball is black, find the probability that it came from
   Box III.

5. Suppose all possible four outcomes $e_1, e_2, e_3, e_4$ of an experiment are equally likely.
   Define the events A, B, C as $A = \{e_1, e_4\}$, $B = \{e_2, e_4\}$, $C = \{e_3, e_4\}$. What can you say about
   dependence or independence of the events A, B, and C?

6. Find the probability that in a throw of two dice, the outcome is even or less than 5.

7. Two newspapers are published in a certain city, it is estimated by a survey that 16% read X,
   14% read Y and 5% read both. Find the probability that a randomly selected person:
   (i) doesn’t read any newspaper
   (ii) reads only Y

8. There are five prisoners, two of whom are to receive a pardon and be released from prison. Two
   white balls and three black balls are placed in a bag with each prisoner drawing a ball from the
   bag. The two prisoners who pick white balls are released. Suppose that the five prisoners take it
   in turns to draw a ball from the bag:
   (a) What is the probability that the first prisoner to draw a ball from the bag is released?
   (b) What is the probability that the second prisoner to draw a ball from the bag is released given
      that the first prisoner to draw a white ball from the bag?
   (c) What is the probability that the third prisoner to draw a ball from the bag is released given
      that the first prisoner to draw a white ball from the bag?
   (d) What is the probability that the last prisoner to draw a ball from the bag is released?

9. Toss a fair coin until the first head appears. Find the probability that an odd number of tosses is
    required to achieve that.
Exercises Solutions

1. If $A$ and $B$ are independent events then so are:
   (a) $A^c$ and $B$            (b) $A^c$ and $B^c$        (c) $A$ and $B^c$
   You’ll be able to see these clearly with the aid of Venn diagrams.
   (a) Since $A^c \cap B$ and $A \cap B$ are mutually exclusive, 
       \[ P(B) = P(B \cap A) + P(B \cap A^c) \] (see axiom 9)
       \[ P(B \cap A^c) = P(B) - P(B \cap A) \]
       but $A$ and $B$ are independent, \[ P(B \cap A) = P(A)P(B) \]
       Therefore $A^c$ and $B$ are independent.
   (b) We use Demorgan’s: $A^c \cap B^c = (A \cup B)^c$
       \[ P(A^c \cap B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c) \]
       \[ P(A^c) + P(B^c) = 1 - P(A) + 1 - P(B) \]
       but $P(A)P(B)$
       \[ P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A)P(B) = 1 - P(A) \]
       Therefore $A^c$ and $B^c$ are independent.
(c)
   \[ P(A) = P(A \cap B) + P(A \cap B^c) \]
   \[ P(A \cap B^c) = P(A) - P(A \cap B) = P(A) (1 - P(B)) \]
   \[ P(A \cap B^c) = P(A)P(B^c) \]
   Therefore $A$ and $B^c$ are independent.

2. (a) \[ P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c)P(C^c) = \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4} \]
   (b) \[ P(A^c \cap B^c \cap C) + P(A^c \cap B \cap C^c) + P(A \cap B^c \cap C^c) \]
       \[ = P(A^c)P(B^c)P(C) + P(A^c)P(B)P(C^c) + P(A)P(B^c)P(C^c) \]
       \[ = \frac{11}{24} \]

3. By Bayes’ rule,
   \[ P(\text{Even} | \text{Blue}) = P(\text{Even}) \frac{P(\text{Blue} | \text{Even})}{P(\text{Even})P(\text{Blue} | \text{Even}) + P(\text{Odd})P(\text{Blue} | \text{Odd})} \]
   \[ = \frac{1}{2} \cdot \frac{1}{2} / \left( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{5} \right) = \frac{5}{9} \]

4. Let $A_1$, $A_2$ and $A_3$ denote the selection of box I, box II and box III respectively, and $B$ denote the event of drawing a black ball.
   Here the events $A_1$, $A_2$ and $A_3$ are exhaustive and mutually exclusive. As the box is selected at random,
   \[ P(A_1) = P(A_2) = P(A_3) = \frac{1}{3} \]
   Also $P(B | A_1) = \frac{5}{8}$, $P(B | A_2) = \frac{6}{8}$ and $P(B | A_3) = \frac{3}{8}$.

(i) The required probability is
   \[ P(B) = \sum_{i=1}^{3} P(A_i)P(B | A_i) = \frac{1}{3} \cdot \frac{5}{8} + \frac{1}{3} \cdot \frac{6}{8} + \frac{1}{3} \cdot \frac{3}{8} = \frac{1}{3} \cdot \frac{14}{12} = \frac{7}{12} \]

(ii) We require $P(A_3/B) = \frac{P(A_3)P(B/A_3)}{\sum_{i=1}^{3} P(A_i)P(B/A_i)}$, by Bayes’ theorem.
    \[ = \frac{3 \cdot \frac{3}{7}}{\frac{7}{12}} = \frac{9}{14} \]
5. Here the sample space is \( S = \{e_1, e_2, e_3, e_4\} \), and \( A = \{e_1, e_4\}, B = \{e_2, e_4\}, C = \{e_3, e_4\} \).

\[ \therefore \quad P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}. \]

Again, \( A \cap B = \{e_4\} \), \( A \cap C = \{e_4\} \), and \( B \cap C = \{e_4\} \).

\[ \therefore \quad P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}. \]

Thus, \( P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B) \),

\[ P(A \cap C) = \frac{1}{4} = P(A) \cdot P(C), \]

and \( P(B \cap C) = \frac{1}{4} = P(B) \cdot P(C). \)

So, the events \( A, B, C \) are pair-wise independent.

We also see that

\[ A \cap B \cap C = \{e_4\}. \]

\[ \therefore \quad P(A \cap B \cap C) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C) = \frac{1}{8}. \]

Hence, the events \( A, B, C \) are not mutually independent, though they are pair-wise independent.

6. There are \( 6^2 = 36 \) elementary events of the experiment, and these are equally likely as the dice are unbiased. Now, let \( E_1 \) and \( E_2 \) denote respectively the events of obtaining an even sum and a sum less than 5. Then, there are 18 elementary events that are favourable to \( E_1 \) and 6 [viz. \( (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2) \)] favourable to \( E_2 \), and 4 [viz. \( (1, 1), (1, 3), (3, 1), (2, 2) \)] favourable to \( E_1 \cap E_2 \).

\[ \therefore \quad P(E_1) = \frac{18}{36}, \quad P(E_2) = \frac{6}{36} \text{ and } P(E_1 \cap E_2) = \frac{4}{36}. \]

We require

\[ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \]

\[ = \frac{18}{36} + \frac{6}{36} - \frac{4}{36} = \frac{20}{36} = \frac{5}{9}. \]
7. Let $A_1$ and $A_2$ denote that the selected person reads X and reads Y, respectively. Since the person is randomly chosen, $P(A_1) = 0.16$, $P(A_2) = 0.14$, $P(A_1 \cap A_2) = 0.05$.

(i) The probability that the chosen person does not read any newspaper is

\[
P(A_1^c \cap A_2^c) = P(A_1^c \cup A_2^c) = 1 - P(A_1 \cup A_2)
\]

\[
= 1 - [P(A_1) + P(A_2) - P(A_1 \cap A_2)]
\]

\[
= 1 - (0.16 + 0.14 - 0.05)
\]

\[
= 0.75.
\]

(ii) In this case, the required probability is

\[
P(A_1^c \cap A_2) = P(A_2) - P(A_1 \cap A_2)
\]

\[
= 0.14 - 0.05
\]

\[
= 0.09.
\]

8. (a) $P(A) = \frac{2}{5}$ . Chance of choosing a white ball.

(b) $P(B|A) = \frac{1}{4}$ .

Given that the first ball is white, it leaves four balls one of which is white.

(c) $P(C|A) = P(C|B \cap A)P(B|A) + P(C|B^c \cap A)P(B^c|A) = \frac{2}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$

(d) $P(E) = \frac{2}{5}$ .

The chances that the last ball is white is equal to the chance that the first ball is white.

9. Suppose the H appears in the first trial, then $p_1 = \frac{1}{2}$

If it appears in the second trial : $TH$, then $p_2 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

If it appears in the third trial : $TT\ H$, then $p_3 = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1/8$

And so on …..

Let A be the event: an even number of tosses is required

\[
P(A) = \sum_{i=even} \frac{1}{2^i} = \frac{1}{2^2} + \frac{1}{2^4} + .......
\]

Which is an infinite geometric progression of ratio $r = \frac{1}{4}$ and first term $a = \frac{1}{4}$

\[
P(A) = \frac{a}{1 - r} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}
\]

Probability an odd number of tosses is required:

\[
P(A^c) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}
\]