1. Alphonse, Bertille and Clarence play a simple game with a fair die. Alphonse tosses the die and observes the number on the uppermost face, which the other two do not see. Bertille tries to guess that number. If she is right, she wins. If she is wrong, then Clarence tries to guess the number and so win the game. If neither Bertille nor Clarence guess correctly, then Alphonse wins the game.

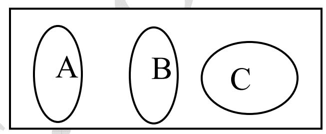
   (a) What is the chance that Clarence wins the game?

   (b) What is the conditional probability that Clarence wins the game given that Alphonse does not win?

S = { 1,2,3,4,5,6}

A : Alphonse wins , B: Bertille wins , P(B) = 1/6 , C : Clarence wins , i.e. Bertille fails

a) $P(B^c) = 1 - P(B) = 1 - 1/6 = 5/6$
   When Bertille fails , Clarence has 5 choices left.
   P(C wins and B fails) = (5/6)(1/5) = 1/6



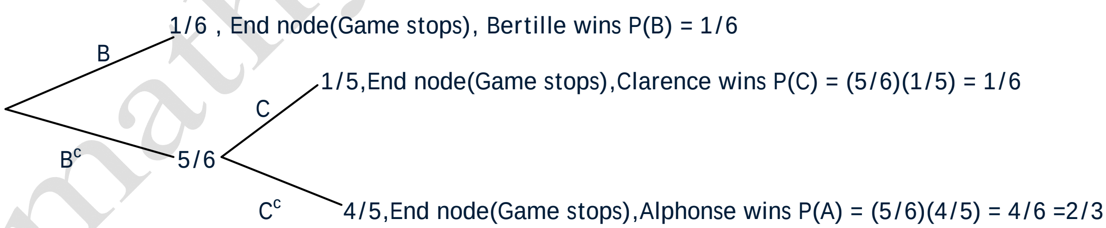
b) $P(C \mid A^c) = \dfrac{P(C \cap A^c)}{P(A^c)}$

c) $C \cap A^c = C \Longrightarrow P(C \mid A^c) = \dfrac{P(C)}{P(A^c)}$

A,B,C are mutually exclusive and exhaustive : P(A) + P(B) + P(C) = 1
$P(B) + P(C) = 1 - P(A) \Longrightarrow 1/6 + 1/6 = 1 - P(A) = P(A^c) = 2/6 = 1/3$

$$P(C \mid A^c) = \frac{P(C)}{P(A^c)} = \frac{1/6}{1/3} = \frac{1}{2}$$

Using Trees :



$P(A^c) = 1 - 2/3 = 1/3$

$$P(C \mid A^c) = \frac{P(C)}{P(A^c)} = \frac{1/6}{1/3} = \frac{1}{2}$$

2. In a large sociology class, 10% of all students get an A grade in their end-of-year examination. 60% of all students had missed no classes. The examiners checked and found that 10% of all students who had got an A grade had missed no classes.

   (a) What is the probability that a student with an A grade had missed no classes?

   (b) What is the probability that a student who missed at least one class did not get an A grade?

   (c) Are the events 'got an A grade' and 'missed no classes' mutually exclusive? Explain.

$P(A) = 0.1$, $P(A^c) = 0.9$, P(Miss no classes)= $P(M^c) = 0.6$, P(Miss classes)=$P(M) = 0.4$

$P(M^c | A) = 0.1$, $P(M | A) = 0.9$

(a) $P(M^c \cap A) = P(M^c | A)\, P(A) = (0.1)(0.1) = 0.01$

(b) $P(A^c | (M^c)^c) = 1 - P(A | M) = 1 - \dfrac{P(A \cap M)}{P(M)} = 1 - \dfrac{P(M | A)\;\; P(A)}{P(M)}$

$$= 1 - \frac{0.9 \times 0.1}{0.4} = 1 - 0.225 = 0.775$$

(c) If A and M^c mutually exclusive then they can not occur at the same time and $A \cap M^c = \phi$ i.e. $P(A \cap M^c) = 0$ since $P(M^c \cap A) = 0.01$, they can not be mutually exclusive.

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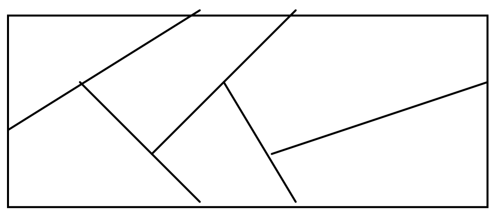
3. State and prove Bayes' Theorem.

If the events B_i , i = 1,……,n form a partition of the sample space Ω , i.e. pair wise disjoint events (pair wise disjoint means any two have no intersection $B_i \cap B_j = \phi$ for all $i \neq j$)

Such that $\bigcup_i^n B_i = \Omega$, then



$$P(B_i | A) = \frac{P(B_i | A) P(B_i)}{\sum_{j=1}^{n} P(A | B_j) P(B_j)}$$

$P(B_i | A) = \dfrac{P(B_i \cap A)}{P(A)} = \dfrac{P(A \cap B_i)}{P(A)}$, but $P(A \cap B_i) = P(B_i | A) P(B_i)$

$$P(B_i | A) = \frac{P(B_i | A) P(B_i)}{P(A)}$$

Since $A \subset \Omega \Rightarrow A = A \cap \Omega \Rightarrow A = A \cap \bigcup_j B_j = \bigcup_j (A \cap B_j)$, distributive law of sets

$$P(A) = P(\bigcup_j (A \cap B_j)) = \sum_j^{1} P(A \cap B_j)$$

Now , for all j , $P(A \cap B_j) = P(A|B_j)P(B_j)$

$$P(B_i|A) = \frac{P(B_i|A)P(B_i)}{P(A)} = \frac{P(B_i|A)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}$$

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4. In six independent tosses of a fair coin, what is the probability that there are at least three successive heads somewhere in the sequence?

P (at least 3 successive heads in 6 tosses) , we need to look at sequences of 3,4,5,6 heads

P(H) = P(T) = ½ , P (any sequence) = $\left(\frac{1}{2}\right)^6$ = ½ x ½ x½ x ½ x½ x ½

The sequence of 3 successive heads occurs in the following manner : ---HHH---
Preceded and proceeded by tails :
Since we have 6 tosses :
6H : HHHHHH (no tails) : P(6H) = 1/64 (1 sequence)
5H : either THHHHH or HHHHHT , (2 sequences) , P(5H) = 2/64
4H : HHHHT (Tor H) , THHHHT, (Tor H)THHHH (6 sequences) , P(4H) = 5/64
3H : HHHT (T or H)(T or H) ,THHHT(T or H), (T or H)THHHT, (T or H), (T or H)THHH
(12 sequences) , P(3H) = 12/64
P (at least 3 successive heads in 6 tosses) = 1/64 + 2/64 + 5/64 + 12/64 = 20/64 = 5/16

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5. If A and B are events such that $P(A|B^c) = 2P(A|B)$ and $P(B^c) = 2P(B)$, show that $P(B^c|A) = 0.8$. (The event B^c is the complement of event B.)

$$P(B^c|A) = \frac{P(B^c \cap A)}{P(A)} = \frac{P(A|B^c)P(B^c)}{P(A)} = \frac{2P(A|B)\times 2P(B)}{P(A)} = \frac{4P(A|B)\times P(B)}{P(A)}$$

P(A) = P(A|B)xP(B) + P(A|B^c)xP(B^c) , but P(A|B^c)xP(B^c) is the numerator in the above fraction
= 4 P(A|B)xP(B) , hence P(A) = P(A|B)xP(B) + P(A|B^c)xP(B^c) = 5 P(A|B)xP(B)

$$P(B^c|A) = \frac{4P(A|B)\times P(B)}{P(A)} = \frac{4P(A|B)\times P(B)}{5P(A|B)\times P(B)} = \frac{4}{5} = 0.8$$

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6. Show that if $P(A \mid B) = P(A \mid B^c)$ then A and B are independent.

To show A and B are independent , we need to show $P(A \cap B) = P(A)P(B)$
P(A) = P(A|B)xP(B) + P(A|B^c)xP(B^c) but P(A|B) = P(A|B^c)
P(A) = P(A|B)xP(B) + P(A|B) xP(B^c) = P(A)(P(B) + P(B^c)) = P(A|B) (1) = P(A|B)

Now $P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)\,P(B) = P(A)\,P(B)$

7. A box of 55 coloured light bulbs consists of 5 blue bulbs, 20 pink bulbs, 20 yellow bulbs and 10 white bulbs. If 4 bulbs are selected at random without replacement, what is the probability that:

   (a) all 4 bulbs are white

   (b) all 4 bulbs are the same colour

   (c) all 4 bulbs are different colours

   (d) no bulb is yellow?

(a) P (wwww) = (10/55)(9/54)(8/53)(7/52) = 5040/ 8185320 = 0.0006

or, P(4W) = $\dfrac{C_4^{10} \times C_0^{5} \times C_0^{20} \times C_0^{20}}{C_4^{55}} = \dfrac{210}{341055} = 0.0006$

(b) P(bbbb) = (5/55)(4/54)(3/53)(2/52) = 120/ 8185320

P(yyyy)=P(pppp) = (20/55)(19/54)(18/53)(17/52) = 116280/ 8185320

P(wwww or bbbb or pppp or yyyy) = 5040/ 8185320 + 120/ 8185320 + 2(116280/ 8185320) = 0.029

(c) P (1w and 1b and 1p and 1y) = ?

P(1w) = $\dfrac{C_1^{10} \times C_3^{45}}{C_4^{55}} = \dfrac{10 \times 14190}{341055} = \dfrac{141900}{341055}$ = 0.416

P(1b) = $\dfrac{C_1^{5} \times C_3^{50}}{C_4^{55}} = \dfrac{5 \times 19600}{341055} = \dfrac{98000}{341055}$ = 0.287

P(1p) = P(1y) = $\dfrac{C_1^{20} \times C_3^{35}}{C_4^{55}} = \dfrac{20 \times 6545}{341055} = \dfrac{130900}{341055}$ = 0.383

P (1w and 1b and 1p and 1y) = (0.416)(0.287)(0.383) (0.383) = 0.017

(d) P(no yellow) = $\dfrac{C_0^{20} \times C_4^{35}}{C_4^{55}} = \dfrac{1 \times 52360}{341055} = 0.153$

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8. Annabel is twice as likely to go shopping as Barbara. Carmel is three times as likely to go shopping when Annabel meets her than she is when Barbara meets her. How much more likely is Annabel to shop than Barbara when each has been met by Carmel?

A: Annabel goes shopping , B: Barbara goes shopping , C : Carmel goes shopping
P(A) = 2P(B)
P(C) is affected by who she meets.
P(A) is independent of who she meets.

This is just a question to make you acquainted with the “independence” concept.