# Previous LSE Questions – Chapter 8

6.(a). Briefly explain each of the following terms:

i. A Random Walk Process

ii. A Queue's Service Mechanism **(4 marks)**

(b) Derive the Poisson distribution from the assumptions of the Poisson process. **(5 marks)**

(c) Matrices $P_1$, $P_2$ and $P_3$ below are transition matrices for Markov chains between states A, B, C and D.

$$P_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

i. Identify all absorbing states, and determine which of the Markov chains are absorbing. [Note: A Markov chain is said to be absorbing if it has at least one absorbing state and if it is possible to go from every state to an absorbing state (not necessarily in one step)]. **(6 marks)**

ii. Draw a network diagram for each transition matrix (with probabilities on the arcs) to illustrate the possible movements. **(3 marks)**

iii. If, for transition matrix $P_3$ one is in state D at time $t = 0$, what is the probability that you will not reach state B at or before $t = 3$ ? **(2 marks)**

2. In a 5 state Markov chain the steps have probabilities 0.3, 0.3 , 0.4 of taking values 1, -1 or 0 respectively. There are reflective barriers at 3 and at -1 and these reflective barriers work such that, if the system reaches a barrier at time $n$, it is reflected back to any of the other 4 states at time period $n$+1 with equal probability.

(a) Write down the transition matrix of the Markov chain. **(3 marks)**

(b) Determine the equilibrium probabilities of being in the various states. [Note: candidates may find it useful to note some symmetry in the situation.]**(10 marks)**

(c) Draw a suitably annotated network diagram to illustrate the above Markov Chain. **(3 marks)**

(d) Create a connectivity matrix to show which states one can move between in exactly one step and use it to determine the number of ways one can move between states in exactly two steps. **(4 marks)**

3a). The following matrix $\{V_{ij} \mid i,j = \text{A,B}\ldots\text{G}\}$ depicts the number of vehicles travelling in a particular hour between cities A,B…G on a road network.

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array}
\begin{array}{c}
\begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\
\begin{pmatrix}
0 & 100 & 300 & 0 & 0 & 0 & 0 \\
200 & 0 & 80 & 0 & 0 & 0 & 0 \\
100 & 180 & 0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 & 120 \\
0 & 0 & 100 & 20 & 0 & 60 & 0 \\
0 & 0 & 0 & 40 & 70 & 0 & 90 \\
0 & 0 & 0 & 100 & 0 & 50 & 0
\end{pmatrix}
\end{array}
$$

i) Draw a network flow diagram showing the vehicle flow between the cities.

(4 marks)

ii) Another matrix $\{T_{ij} \mid i,j = \text{A,B}\ldots\text{G}\}$ depicts the time (in minutes) taken for travelling between $i$ and $j$ where

$$T_{ij} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{pmatrix} 0 & 60 & 40 & 0 & 0 & 0 & 0 \\ 50 & 0 & 50 & 0 & 0 & 0 & 0 \\ 10 & 50 & 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 130 & 50 \\ 0 & 0 & 30 & 90 & 0 & 40 & 0 \\ 0 & 0 & 0 & 120 & 20 & 0 & 30 \\ 0 & 0 & 0 & 70 & 0 & 25 & 0 \end{pmatrix} \end{array}$$

Use matrix algebra to determine a matrix of times to complete a return trip from $i$ to $j$ and back again. (4 marks)

3b)i) Describe and solve the general 'Gambler's Ruin' problem. (9 marks)

ii) Assume that:

Each player bets and stands to win \$10 on each play;

Initially player A has \$40 and player B has \$30;

That the odds are fair such that for each play A will win with probability 0.5 and B will win with probability 0.5, and

That the game continues until one of the players is ruined.

What is the probability that B will eventually be ruined? (3 marks)

7. An industrial company negotiates contracts which may run for up to 3 months or be cancelled at 1 week's notice. At the end of any given month, $t$, there are $x(t)$ contracts less than 1 month old, $y(t)$ contracts between 1 and 2 months old and $z(t)$ contracts between 2 and 3 months old.

New contracts arise and some of the existing ones either finish or are cancelled in such a way that the age and number of contracts at the end of one month are dependent upon the age and number at the end of the previous month. The following matrix equation best fits the situation:

$$\begin{pmatrix} x(t+1) & y(t+1) & z(t+1) \end{pmatrix} = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}.P$$

where $P$ is known as the *projection matrix* and is defined to be :

$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.5 & 0 & 0.5 \\ 0.4 & 0.6 & 0 \end{pmatrix}$$

[Use matrix algebra wherever possible in answering the following question].

i) If $x(0) = 10$, $y(0) = 80$ and $z(0) = 50$ determine the number of contracts in each age category at the end of each of the next two months. (6 marks)

ii) Determine the long-run equilibrium values for $x$, $y$ and $z$. (4 marks)

iii) By inverting $P$ find $x(t)$, $y(t)$ and $z(t)$ if $x(t+1) = 28$, $y(t+1) = 28$ and $z(t+1) = 14$. (10 marks)