3.1 The Poisson Process

Introduction

Markov Processes: These differ from Markov chains in one major aspect: time now becomes a continuous quantity (i.e. **now it is time as we know it)**, The simplest example of a Markov process is the so called Poisson.

The Poisson process

It expresses the probability of a number of events occurring in a fixed time if these events occur with a known average rate, and are independent of the time since the last event. A number of discrete occurrences (sometimes called "arrivals") that take place during a time-interval of given length. The probability that there are exactly *k* occurrences is a poisson distribution of **rate λ**:

$$P(\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

λ is a positive real number, equal to the expected number of occurrences that occur during the given interval. For instance, if the events occur on average every 4 minutes, and you are interested in the number of events occurring in a 10 minute interval, you would use as model a Poisson distribution with λ = 2.5.

Examples: Calls arriving at a telephone exchange in a day.

The number of people joining a queue in an hour.

A stochastic process $N(t)$; $t \geq 0$ is a (time-homogeneous, one-dimensional) Poisson process if,

-The number of events occurring in two disjoint (**non-overlapping**) subintervals are independent random variables*(see No. 2 in the assumptions below).*Arrivals are *memoryless* i.e. independent of what has happened before.

Pr(k arrivals in time t) = P(λt) = $\frac{e^{-\lambda t}(\lambda t)^k}{k!}$;

Pr(k arrivals in [0,t]) = P(λt) = $\frac{e^{-\lambda t}(\lambda t)^k}{k!}$

i.e. a Poisson distribution of parameter λt .

Assumptions of the Poisson process:

If {N(t) ; t≥ 0 } is poisson process of parameter λ then :

1. N(0) = 0
2. For any $t_0 = 0 < t_1 < t_2 < \ldots < t_n$,the process increments

   $N(t_1) - N(t_0)$: the number of events in $(0,t_1]$

   $N(t_2) - N(t_1)$: the number of events in $(t_1,t_2]$

   etc……

   are independent random variables.
3. The number of events occurring in the time interval (s , t +s]

   is a Poisson(λt); i.e. Pr[N(s+t) – N(s)] ~ P(λt)
4. N(t) ~ P(λt) ; i.e. Pr(k arrivals in time t) = P(λt)

Example:

A company expects on average, four of its trucks will break down in a one-month Period. Assuming a Poisson distribution is appropriate, what is the probability that exactly four trucks break down in a month? in a two month period?

If X is the number of trucks which break down in a month then as given

X ~ P(4) = $\frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-4}4^4}{4!}$ = 0.195 i.e. 19.5 % of months would have 4 trucks breakdown.

If the rate of breakdowns in a month is 4 ,then the rate of breakdowns in two months is 2x4 = 8 ; hence the No. of breakdowns in two months ~ P(8) : = $\frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-8}8^4}{4!}$ **= 0.0572**

EXAMPLE 1.2.2. *Fax messages arrive at an office according to a Poisson process at mean rate three per hour.*

(a) *What is the probability that exactly two messages are received between 9.00 and 9.40?*
(b) *What is the probability that no messages arrive between 10.00 and 10.30?*
(c) *What is the probability that not more than three messages are received between 10.00 and 12.00?*

SOLUTION. Let an hour be the unit of time that we work with. Thus we have a Poisson process of rate $\lambda = 3$. The number of messages in time t is thus $\text{Pois}(3t)$ distributed.

(a) 9.00 to 9.40: the number of messages has distribution $\text{Pois}\left(3 \times \frac{2}{3}\right) = \text{Pois}(2)$, so

$$P(2 \text{ messages}) = e^{-2}\frac{2^2}{2!} \simeq 0{\cdot}271.$$

(b) 10.00 to 10.30: the number of messages has distribution $\text{Pois}(3 \times 0{\cdot}5) = \text{Pois}(1{\cdot}5)$, so

$$P(0 \text{ messages}) = e^{-1\cdot 5} \simeq 0{\cdot}223.$$

(c) 10.00 to 12.00: the number of messages, N say, has distribution $\text{Pois}(3 \times 2) = \text{Pois}(6)$, so

$$\begin{aligned} P(N \le 3) &= P(N=0) + P(N=1) + P(N=2) + P(N=3) \\ &= e^{-6}\frac{6^0}{0!} + e^{-6}\frac{6^1}{1!} + e^{-6}\frac{6^2}{2!} + e^{-6}\frac{6^3}{3!} \\ &= e^{-6}(1 + 6 + 18 + 36) = 61e^{-6} \simeq 0{\cdot}151. \end{aligned}$$

Distribution of T

We will consider two random variables:

- $X(t)$, the number of points that occur in $(0, t]$;
- T, the time until the first point occurs.

T is the time until the next point of the process from a given starting time. As the process is 'memoryless' it does not matter when this starting time is. T is a continuous random variable, unlike $X(t)$, which is discrete.

Consider the event $\{T > t\}$, i.e. the event that T is larger than some specified t. This is identical to $\{X(t) = 0\}$, the event that there is no point of the process up to time t. Thus

$$P(T > t) = P(X(t) = 0) = e^{-\lambda t}.$$

The distribution function of T is thus $P(T \leq t) = 1 - e^{-\lambda t}$, i.e. T has an exponential distribution with parameter λ.

EXAMPLE 1.3.1. *In Example 1.2.2, what is the probability that the first message after 10.00 occurs by 11.00?*

SOLUTION. We have a Poisson process of rate 3 per hour and want the probability that the interval from our chosen starting time until the first point of the process is at most 1 hour, so

$$P(T \leq 1) = 1 - e^{-3\times 1} \doteq 0{\cdot}9502.$$

The pooled Poisson process

Suppose that a bank branch has k service points in use and operates with a single long queue, so that when a server finishes serving a customer the person at the head of the queue replaces that customer. Further suppose that each service time is exponential with parameter λ, independently of all other service times.

The time that a person spends at the head of the queue depends upon k Poisson processes each with parameter λ.

Let T be the length of the waiting time for the person at the front of the queue until service starts. If T_i is the time until the person currently at server i is served, then $T = \min(T_1, \ldots, T_k)$. So

$$\begin{aligned} P(T > t) &= P(\min(T_1, \ldots, T_k) > t) \\ &= P(T_1 > t, T_2 > t, \ldots, T_k > t) \\ &= P(T_1 > t)P(T_2 > t) \cdots P(T_k > t) \\ &= e^{-\lambda t} \times \cdots \times e^{-\lambda t} = e^{-k\lambda t}, \end{aligned}$$

i.e. $P(T \leq t) = 1 - e^{-k\lambda t}$. Thus T is exponential with parameter $k\lambda$. Equivalently T is the time to the first point of a Poisson process of rate $k\lambda$.

Combined processes : *If k independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$ occur simultaneously, the combined points follow a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.*

EXAMPLE 1.4.3. *In the same office as in the previous two examples, telephone messages arrive at the mean rate of six per hour.*

(a) Find the probability that exactly two messages (phone or fax) are received between 9.00 and 9.40.

(b) Find the probability that the first message after 10.00 occurs before 10.10.

SOLUTION.

(a) We have two independent Poisson processes with rates 3 and 6 respectively, i.e. the pooled process has rate $3+6=9$. So in a 40-minute period, the number of calls follows a Poisson distribution with parameter

$$9 \times \frac{2}{3} = 6.$$

Therefore

$$P(X = 2) = e^{-6}\frac{6^2}{2!} \doteq 0{\cdot}0446.$$

(b)

$$\begin{aligned} &P(\text{first message after 10.00 is before 10.10}) \\ &\quad = P(T \leq 1/6) \\ &\quad = 1 - e^{-9\times 1/6} = 1 - e^{-3/2} \doteq 0{\cdot}7769. \end{aligned}$$

Proof of the Poisson Distribution

Remark: define o(δt) as $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$

Suppose Pr(**One** event in (t + δt)) = λ δt + o(δt) , where o(δt) is negligably small when divided by δt.

Here o(δt) represents terms which approach 0 more rapidly than does δt as $\delta t \to 0$.

Let $p_n(t)$ be the the probability of n events in time t then :

$p_n(t + \delta t) = (1 - \lambda \delta t)p_n(t) + \lambda \delta t\, p_{n-1}(t) + o(\delta t)$

Rearranging : $\frac{p_n(t+\delta t) - p_n(t)}{\delta t} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(\delta t)}{\delta t}$

as $\delta t \to 0$

$\frac{dp_n(t)}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t)$ for $n \geq 1$ and $\frac{dp_0(t)}{dt} = -\lambda p_0(t)$

Solving these equations :

$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ the Poisson distribution with parameter λt

Remark: to explain what is going above, I found that I need to write 3 pages!! Take it as it is and pretend that you know what is going on.

Birth – Death Processes

Simple Birth process

Consider a population where each individual alive in the population generates further offspring according to a Poisson process at rate β

assume that the initial population size is x_0 and that there are no deaths, so that the population increases with time.

If the size of the population at time t is x, then

we have x different Poisson processes each with rate β. When considering the next birth we thus have a pooled Poisson process with rate $x\beta$ (until the next arrival, when this becomes a Poisson process with rate $(x+1)\beta$).

The distribution of $X(t)$ is given by

$$p_x(t) = \binom{x-1}{x_0-1} e^{-\beta t x_0}(1-e^{-\beta t})^{x-x_0} \quad (x = x_0, x_0+1, \ldots).$$

This is sometimes called a 'negative binomial distribution with scale parameter $e^{-\beta t}$ and index parameter x_0'. However the name is better used for the distribution of $X(t)-x_0$, taking values in $\{0, 1, 2, \ldots\}$.

If $x_0 = 1$, the distribution of $X(t)$ becomes

$$p_x(t) = e^{-\beta t}(1-e^{-\beta t})^{x-1} \qquad (x = 1, 2, \ldots),$$

sometimes called the 'geometric distribution with parameter $e^{-\beta t}$'. However, again the name is better used for the distribution of $X(t) - 1$, taking values in $\{0, 1, 2, \ldots\}$.

EXAMPLE 2.1.2. *A population starts at time 0 with a single individual. Let the birth rate be two per week.*

(a) What is the probability that after three weeks there are exactly two individuals?

(b) What is the probability that after one week there are between two and four individuals (inclusive)?

SOLUTION.

(a) $x_0 = 1$, $\beta = 2$ per week, and $t = 3$, i.e.

$$\begin{aligned} p_2(3) &= \binom{1}{0} e^{-2\times 3}(1-e^{-2\times 3})^1 \\ &= e^{-6}(1-e^{-6}) \doteq 0{\cdot}00247. \end{aligned}$$

(b) $x_0 = 1$, $\beta = 2$, $t = 1$, so

$$\begin{aligned} &P(2 \le X \le 4) \\ &\quad = P(X=2) + P(X=3) + P(X=4) \\ &\quad = e^{-2}(1-e^{-2}) + e^{-2}(1-e^{-2})^2 + e^{-2}(1-e^{-2})^3 \\ &\quad \doteq 0{\cdot}3057. \end{aligned}$$

The pure Death process

We consider a population in which there are no births, just deaths. Observations start with x_0 individuals alive at time 0—these individuals die independently of each other, and eventually the population dies out completely.

In a similar way to the simple birth process, we assume that the probability of an individual dying in time interval $(t, t+\delta t]$ is $\nu\delta t + o(\delta t)$.

Some questions of interest are

- What is the distribution of the population size at time t?
- How long does it take the population to die out?

This model is approached best by considering every individual separately. The probability that a given individual is alive at time t, which we label $P_a(t)$, is found as follows. The probability that the individual, if alive at time t, is still alive at time $t+\delta t$ is one minus the probability that it dies in this interval, so that

$$P(\text{alive at } t+\delta t|\text{alive at } t) = 1 - \nu\delta t + o(\delta t);$$
$$\therefore \qquad P_a(t+\delta t) = P_a(t)\big(1 - \nu\delta t + o(\delta t)\big);$$
$$\therefore \qquad P_a'(t) = -\nu P_a(t).$$

This differential equation has solution $P_a(t) = Ae^{-\nu t} = e^{-\nu t}$ since the individual is alive at time 0 with probability 1. We can use the binomial theorem to deduce that the probability that j individuals are still alive at time t is given by

$$p_j(t) = \binom{x_0}{j}(e^{-\nu t})^j(1 - e^{-\nu t})^{x_0 - j}.$$

In particular the probability that the population is extinct by time t is

$$p_0(t) = (1 - e^{-\nu t})^{x_0}.$$

EXAMPLE 2.2.1. *A population starts at time 0 with 4 individuals. The population follows a pure death process at a rate of 1 every 2 days.*

(a) Find the probability that there is exactly one individual alive after a week.
(b) Find the probability that the population has died out after a week.
(c) Find the probability that the population has died out after two weeks, given that the total number of survivors after one week was 2.

SOLUTION. $x_0 = 4$, $\nu = 0{\cdot}5$ per day.

(a) $t = 7$, so

$$p_1(7) = \binom{4}{1} e^{-3\cdot5}(1 - e^{-3\cdot5})^3 \doteq 0{\cdot}1102.$$

(b) Again $t = 7$, so

$$p_0(7) = (1 - e^{-3\cdot5})^4 \doteq 0{\cdot}8846.$$

(c) The process is memoryless, so that

$$\begin{aligned} &P(0 \text{ after 2 weeks}|2 \text{ after 1 week}) \\ &\quad = P(0 \text{ after 1 week}|2 \text{ after 0 weeks}) \\ &\quad = (1 - e^{-3\cdot5})^2 \doteq 0{\cdot}9405. \end{aligned}$$

Birth-Death Process

In the simple birth process, each individual gives birth at rate β, so that when the population is of size x, the birth rate is βx. In the pure death process individuals die at rate ν, so that the death rate is νx. We wish to find an expression for $X(t)$, the number of individuals alive at time t.

- In the classical gambler's ruin problem, there are two players with £m between them, so that the probability that our gambler wins is the probability that (s)he reaches m before 0, which, starting at $j \le m$, is

$$\begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^j}{1-\left(\frac{q}{p}\right)^m} & \text{if } p \ne q, \\ \dfrac{j}{m} & \text{if } p = q. \end{cases}$$

This corresponds exactly to the probability that the size of our population reaches $m \ge x_0$ individuals at some point (before possibly becoming extinct), which thus has probability

$$\begin{cases} \dfrac{1-\left(\frac{\nu}{\beta}\right)^{x_0}}{1-\left(\frac{\nu}{\beta}\right)^m} & \text{if } \beta \ne \nu, \\ \dfrac{x_0}{m} & \text{if } \nu = \beta. \end{cases}$$

EXAMPLE 3.5.1. *If a simple birth-death process starts with $x_0 = 5$ individuals, what is the probability that it reaches 10 given that it becomes extinct, in the cases*

(a) $\beta = 4$, $\nu = 6$?

SOLUTION.

(a)

$$P(\text{reaches } 10) = \frac{1-\left(\frac{6}{4}\right)^5}{1-\left(\frac{6}{4}\right)^{10}} \doteq 0{\cdot}1164.$$

The process is certain to become extinct, so that

$$\begin{aligned} P(\text{reaches } 10|\text{becomes extinct}) &= P(\text{reaches } 10) \\ &\doteq 0{\cdot}1164. \end{aligned}$$

Queuing Theory

In many aspects of everyday life we encounter queues: telephone queueing systems (becoming increasingly common), bank/shop queues, traffic jams, queues of aircraft circling an airport, etc. Patients have to 'queue' to wait for an operation. Sometimes we can see the size of the queue, sometimes not. It may or may not be possible to make a good guess at the length of the time to wait. We may not have to wait at all, or the queue could be so long that we decide to give up and try again some other time.

From the point of view of the server/shop owner it might be important to consider breaks (when nobody is queueing) or the possibility that arrivals come too quickly to be served (i.e. the queue gets longer and longer and/or people give up).

The queueing process is unpredictable in that the rate of customer arrivals and the time it takes for a customer to be served are both random. It is of interest to model queues as a random process because they are common and because some of the parameters can be controlled, e.g. by varying the number of servers, so it would be useful to understand the consequences of such variations.

There are three features of a queue which we shall consider (in reality, of course, there are many):

- The arrival mechanism—how do customers arrive, singly or in groups, randomly or by appointment?
- The service time—constant or random, what distribution?
- The number of servers.

Another question is that of queue discipline. Are customers served in the actual order by which they arrive? If there is a single queue in a bank, the answer is usually yes, but in a pub there is often a random element (which person the barman sees first) or a not-so-random element (Joe might be served earlier because he's in there every day). We shall assume that customers are served *strictly in the order of arrivals.*

If there is more than one server, we assume a central queueing system, so that customers move forward as servers become free.

We shall consider only queueing models formed by varying the three features mentioned above. In general we assume that arrivals occur singly and at random.

A queue will be characterised as follows to describe its three features. Firstly we specify the inter-arrival time, e.g. Poisson process/exponential, written as M (for Markovian), deterministic (fixed interval) D, general (unspecified) G, etc. Similarly we specify the service time. Finally we state the number of servers.

EXAMPLE 4.1.1. *A local bank has two cash dispensers. A customer arrives and joins a central queue for both machines (or uses a machine if one is free). Assuming it always takes the same time to use a machine, and that customers arrive at random, specify the queueing system.*

SOLUTION. 'Arrive at random' means arrivals are a Poisson process, 'M'. Service time constant: 'D'. Two servers. The queue is M/D/2. □

The simple queue

Customers arrive singly, independently of one another at a service point. We assume that they arrive as a Poisson process, with arrival rate λ. This is equivalent to the distribution of the inter-arrival time being exponential with parameter λ.

This is a reasonable model for banks and supermarkets, but not for cinemas, where arrivals cluster near the time when films start.

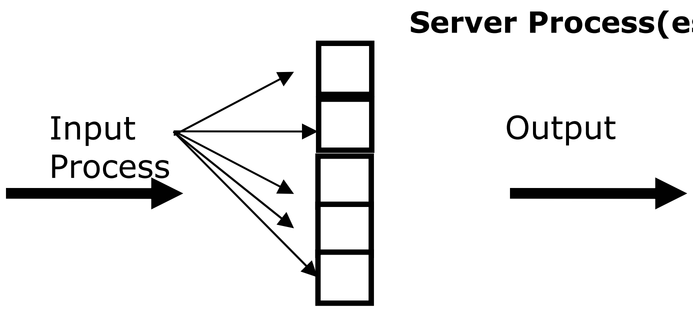
If the server is free, the customer goes straight to the server and is served immediately. Otherwise they join the end of the queue. Customers are served in the order of their arrival. We define the following.

- The *service time* is the time the customer takes to be served once he/she reaches the server.
- The *waiting time* is the time for all customers ahead of the new arrival to be served (including the one at the server).
- The *queueing time* = waiting time + service time.

Suppose that the service time is distributed exponentially, with parameter μ (i.e. customers leave according to a Poisson process if the queue is non-empty). μ is the *service rate* of the queue.

For our simple model, the queue specification is M/M/1. This is called the *simple queue*.

Queuing Theory deals with systems of the following type:



Typically we are interested in how much queuing occurs or in the delays at the servers.

A standard notation is used in queuing theory to denote the type of system we are dealing with.

Typical examples are:

- M/M/1 Poisson Input/Poisson Server/1 Server
- M/G/1 Poisson Input/General Server/1 Server
- D/G/n Deterministic Input/General Server/n Servers
- E/G/∞ Erlangian Input/General Server/Infinite Servers

The *first* letter indicates the *input process*(**M = Memoryless = Poisson**), the *second* letter is the *server* process and the *number* is the **number of servers**.

The simplest queue is the M/M/1 queue

Memoryless=Poisson/Memoryless=Poisson/1 server

Example 1

Cars pass a certain tree on a quiet country road at an average rate of one every two minutes.

(a) What is the probability that exactly five cars pass the tree in ten minutes?

(b) What is the probability that exactly five cars pass the tree in ten minutes, given that in the first five of these minutes one car passes the tree?

(c) What is the probability that in all of the periods 10.00–10.02, 10.02–10.04, 10.04–10.06, 10.06–10.08, 10.08–10.10, at least one car passes the tree?

(d) Would a Poisson process be a good model for a busy road? Explain.

Solution

The Poisson process has rate $\lambda = 0{\cdot}5$ per minute.

(a) $\lambda t = 0{\cdot}5 \times 10 = 5$, so

$$P(X(10) = 5) = e^{-5}\frac{5^5}{5!} = \frac{625}{24}e^{-5} \simeq 0{\cdot}1755.$$

(b) We want the conditional probability that in the latter five minutes exactly four cars pass the tree, given that 1 car passes in the first five minutes. However, as the numbers of points of the process in disjoint time-intervals are independent random variables, the conditional probability here equals the unconditional probability, i.e. we just want the probability that in the latter five minutes exactly four cars pass the tree. As $\lambda t = 0{\cdot}5 \times 5 = 2{\cdot}5$, this is

$$P(X(5) = 4) = e^{-2\cdot 5}\frac{(2{\cdot}5)^4}{4!} = \frac{625}{384}e^{-2\cdot 5} \simeq 0{\cdot}1336.$$

(c) The probability that at least one car passes the tree in a given 2-minute period is $1 - e^{-0\cdot 5\times 2} = 1 - e^{-1}$. As the numbers passing the tree in separate periods are independent, the probability that at least one car passes in each of 5 non-overlapping two-minute periods is $(1 - e^{-1})^5 \simeq 0{\cdot}6321^5 = 0{\cdot}1009$.

(d) No. Cars prevent other cars passing for a short time-period, until the road-space occupied by the first car is clear. So the assumption of independent numbers of points occurring in disjoint time-intervals is violated.

Example 2

A certain large pub has three entrances: the front, the back and the side doors. The landlord thinks that during lunchtime customers arrive at rate one per minute through the front door, one every two minutes through the back door and one every three minutes through the side door, and models this by a Poisson process for each entrance. The pub opens at 12.00. What is the probability of each of the following events?

(a) No customers arrive through the front door by 12.06.

(b) No customers arrive through any door by 12.06.

(c) Exactly five customers arrive in the pub by 12.06, given that exactly two came through the front door.

(d) At least two customers enter by each door by 12.06.

(e) The second customer to enter the pub comes in through the back door.

Solution

(a) The number arriving through the front door in 6 minutes is Poisson of parameter $1 \times 6 = 6$, so the probability that none arrive is $e^{-6} \simeq 0{\cdot}00248$.

(b) The number arriving through any door is a pooled Poisson process of rate

$$1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

per minute, so the number arriving in the 6-minute period is Poisson of parameter

$$\frac{11}{6} \times 6 = 11,$$

and the probability that none arrive is $e^{-11} \simeq 0{\cdot}0000167$.

(c) We want the conditional probability that 3 customers enter through the back or side doors by 12.06, given that 2 enter through the front door. As the numbers entering through the three doors are independent, the conditional probability equals the unconditional probability. Entry through the back or side doors is a pooled Poisson process of rate

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

per minute, so the number arriving in the 6-minute period is Poisson of parameter

$$\frac{5}{6} \times 6 = 5,$$

and the probability that exactly 3 arrive is

$$\frac{5^3}{3!}e^{-5} = \frac{125}{6}e^{-5} \simeq 0{\cdot}140.$$

(d) The numbers entering by the three doors in the 6-minute period are independent Poisson random variables of parameters

$$1 \times 6 = 6, \quad \frac{1}{2} \times 6 = 3, \quad \frac{1}{3} \times 6 = 2$$

respectively, so the probability that all these random variables are at least 2 is

$$\begin{aligned}(1 - e^{-6} - 6e^{-6})(1 - e^{-3} - 3e^{-3})(1 - e^{-2} - 2e^{-2}) &= (1 - 7e^{-6})(1 - 4e^{-3})(1 - 3e^{-2}) \\ &\simeq 0{\cdot}9826 \times 0{\cdot}8009 \times 0{\cdot}5940 = 0{\cdot}467.\end{aligned}$$

(e) The probability that the second (or any other) customer enters through the back door is

$$\frac{\frac{1}{2}}{1 + \frac{1}{2} + \frac{1}{3}} = \frac{1}{2} \times \frac{6}{11} = \frac{3}{11} \simeq 0{\cdot}273.$$