International Institute for Technology and Management

Tutoring Sheet #5 - Solution

Unit 76: Management Mathematics –Difference Equations

1. Solve the following Difference Equations:

a. $y_t = 3y_{t-1} + 4$; $y_0 = 2$; $y_t = 3^t y_0 + 4(\frac{1-3^t}{1-3}) = 2(3^t) - 2(1-3^t)$

b. $y_{t+3} = 2y_{t+2} - 5$; $y_t = 2^{t-2} y_2 - 5(\frac{1-2^{t-2}}{1-2}) = 2^{t-2} y_2 + 5(1-2^{t-2})$

c. $y_{t+1} = y_t + 13$; $y_0 = 5$; $y_t = y_0 + 13t = 5 + 13t$

2. Solve the following difference equations and describe the solution series:

a. $y_{t+2} - y_{t+1} - 2y_t = 3$ with $y_0 = 2$; $y_1 = 2$

The auxiliary equation : $r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0$

$r = -1$; $r = 2$ two distinct real roots.

The complementary function : $y_c = A(-1)^t + B(2)^t$

The equation has the form : $y_{t+2} + ay_{t+1} + by_t = 0$

Where $a + b = -1 - 2 = -3 \neq -1$

For a particular solution , $y_p = \frac{c}{1+a+b} = -3/2$

The general solution : $y_t = y_c + y_p = A(-1)^t + B(2)^t - 3/2$

Now use $y_0=2$, $y_1= 2$ to find A and B

$y_0=2 \Rightarrow A(-1)^0 + B(2)^0 - 3/2 = 2 \Rightarrow A + B = 7/2$

$y_1= 2 \Rightarrow A(-1)^1 + B(2)^1 - 3/2=2 \Rightarrow -A + 2B = 7/2$

Solving for A and B simultaneously: $A = 7/6$; $B = 7/3$

The general solution : $y_t = (7/6)(-1)^t + (7/3)(2)^t - 3/2$

b. $4y_{t+2} + 4y_{t+1} + y_t = 2+3t$ with $y_1 = 1$; $y_2 = 2$

The auxiliary equation : $4r^2 + 4r + 1 = 0 \Rightarrow (2r + 1)(2r + 1) = 0$

$r = -1/2$ two equal real roots.

The complementary function : $y_c = (A + Bt)(-1/2)^t$

For a particular solution , $y_p = C + Dt$ substitute this in the original Equation : $4y_{t+2} + 4y_{t+1} + y_t = 2 + 3t$

$\Rightarrow 4[C + D(t+2)] + 4[C + D(t+1)] + C + Dt = 2 + 3t$

$\Rightarrow (4D + 4D + D)t + 4C + 4C + C + 8D + 4D = 2 + 3t$

$\Rightarrow 9Dt + 9C + 12D = 2 + 3t$ equating coefficients of t and the constant terms: $9D = 3$; $D = 1/3$; $9C + 12(1/3) = 2$; $C = -2/9$

Hence $y_p = (1/3)t - 2/9$

The general solution : $y_t = y_c + y_p = (A + Bt)(-1/2)^t + (1/3)t - 2/9$

Now use $y_1=1$, $y_2= 2$ to find A and B :

$y_1=1 \Rightarrow (A + B(1))(-1/2)^1 + (1/3)(1) - 2/9 = 1 \Rightarrow (A+B)(-1/2) = 8/9$

$y_2= 2 \Rightarrow (A + B(2))(-1/2)^2 + (1/3)(2) - 2/9 = 2 \Rightarrow A/4+B/2 = 14/9$

Solving for A and B simultaneously: $A = -88/9$; $B = 8$

The general solution : $y_t = (-88/9 + 8t)(-1/2)^t + (1/3)t - 2/9$

c. $y_t + 4y_{t-2} = 15$ with $y_0 = 12$; $y_1 = 11$

The auxiliary roots : $r^2 + 4 = 0 \Rightarrow r = -2i$, $r = +2i$

The complementary function: $y_c = (\sqrt{4})^t (A\cos\alpha t + B\sin\alpha t)$

$\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$; $y_t = 2^t\left(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t\right)$

The equation has the form : $y_t + ay_{t-1} + by_{t-2} = 0$

Where $a + b = 0+4 = 4 \neq -1$

For a particular solution , $y_p = \frac{c}{1+a+b} = 3$

The general solution: $\mathbf{y_t = y_c + y_p} = 2^t\left(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t\right) + 3$

$y_0 = 12 \Rightarrow 2^0(A\cos 0 + B\sin 0) + 3 = 12 \Rightarrow A+3 = 12 \Rightarrow A = 9$

$y_1 = 11 \Rightarrow 2^1\left(A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}\right) + 3 = 11 \Rightarrow 2B+3 = 11 \Rightarrow B = 4$

The general solution :

$$\mathbf{y_t} = 2^t\left(9\cos\frac{\pi}{2}t + 4\sin\frac{\pi}{2}t\right) + 3$$

d. y_{t+2} +4 y_{t+1} + 5 y_t = 20

The auxiliary equation : r^2 +4r +5 = 0 $\Rightarrow$ 2 complex roots -2 $\pm$ 4i

The complementary function: $y_c = (\sqrt{5})^t(A\cos\alpha\, t + B\sin\alpha\, t)$

$\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(\frac{-4}{2\sqrt{5}}) = 153.43° \times \frac{\pi}{180} = 0.85\pi$

; $y_t = (\sqrt{5})^t(A\cos 0.85\pi t + B\sin 0.85\pi t)$

The equation has the form : $y_{t+2} + ay_{t+1} + by_t = 0$

Where a + b = 4+5 = 9 $\neq -1$

For a particular solution , $y_p = \frac{c}{1+a+b}$ = 2

The general solution: $\mathbf{y_t = y_c + y_p}$

$\mathbf{y_t} = (\sqrt{5})^t(A\cos 0.85\pi t + B\sin 0.85\pi t) + 2$

e. y_t +4 y_{t-2} = 13 - 5t with y_0 = 6 ; y_1 = 8

The auxiliary roots : $r^2 + 4 = 0 \Rightarrow$ r = $-2i$, r = $+2i$

The complementary function: $y_c = (\sqrt{4})^t(A\cos\alpha\, t + B\sin\alpha\, t)$

$\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(0) = \frac{\pi}{2}$; $y_t = 2^t(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t)$

For a particular solution , y_p = C + Dt substitute this in the original Equation : $y_t + 4y_{t-2}$ = 13 - 5t

C+Dt + 4[C+D(t-2)] = 13 -5t $\Rightarrow$ 5Dt +5C – 8D = 13 -5t

$\Rightarrow$ D = -1 ; C =1 $\Rightarrow$ y_p = 1 - t

The general solution: $\mathbf{y_t = y_c + y_p}$ = $2^t(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t)$ +1- t

y_0 = 6 $\Rightarrow 2^0(A\cos 0 + B\sin 0)$ +1- 0=6 $\Rightarrow$ A+1 =6 $\Rightarrow$ A = 5

y_1 = 8 $\Rightarrow 2^1(A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2})$ +1-1 =8 $\Rightarrow$ 2B = 8 $\Rightarrow$ B= 4

The general solution :

$$\mathbf{y_t} = 2^t(5\cos\frac{\pi}{2}t + 4\sin\frac{\pi}{2}t) + 1 - t$$

3. Consider the following difference equation:
$y_t + 4y_{t-2} = 22 + 5t$ with $y_0 = 1$; $y_1 = 3$
Solve the above difference equation for y ,graph the Solution and describe the graph in words.

The auxiliary roots : $r^2 + 4 = 0 \Rightarrow r = -2i$, $r = +2i$

The complementary function: $y_c = (\sqrt{4})^t (A\cos\alpha\, t + B\sin\alpha\, t)$

$\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(0) = \frac{\pi}{2}$; $y_t = 2^t(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t)$

For a particular solution , $y_p = C + Dt$ substitute this in the original Equation : $y_t + 4y_{t-2} = 13 - 5t$

$C+Dt + 4[C+D(t-2)] = 22 + 5t \Rightarrow 5Dt + 5C - 8D = 22 + 5t$
$\Rightarrow D = 1$; $C = 6 \Rightarrow y_p = 6 + t$

The general solution: $\mathbf{y_t = y_c + y_p} = 2^t(A\cos\frac{\pi}{2}t + B\sin\frac{\pi}{2}t) + 6 + t$

$y_0 = 1 \Rightarrow 2^0(A\cos 0 + B\sin 0) + 6 + 0 = 1 \Rightarrow A + 6 = 1 \Rightarrow A = -5$

$y_1 = 3 \Rightarrow 2^1(A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}) + 6 + 1 = 3 \Rightarrow 2B + 7 = 3 \Rightarrow B = -2$

The general solution :

$$\mathbf{y_t} = 2^t(-5\cos\frac{\pi}{2}t - 2\sin\frac{\pi}{2}t) + 6 + t$$

since the auxiliary roots are complex, it is oscillating.
Since |2| > 1 , it is divergent.