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Unit 76: Management Mathematics

Handout #7a

Applications of Matrices

Dowling ET ,Schaum's Outline series: Introduction to mathematical economics Chapters 10, 11

Topic	Interpretation
Matrix Definition A matrix is an array of numbers: **A =** $\begin{pmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & ...... & a_{mn} \end{pmatrix}$ Matrices are denoted by capital letters : **A,B,C,.....** Matrix size or rank is determined by the number of rows × the number of columns it has. We say **A** has **m** rows and **n** columns or it is an **m**×**n** matrix.	*Example1:* **A =** $\begin{pmatrix} 2 & 0 \\ -5 & 3 \end{pmatrix}$ is **2** × **2** **B =** $\begin{pmatrix} 3 & 0 & -1 \\ 6 & 8 & 2 \\ 1 & 0 & 7 \\ -5 & -1 & 4 \end{pmatrix}$ is **4** × **3** **C =** $\begin{pmatrix} 1 & 6 & 5 & -2 & 3 \end{pmatrix}$ is **1** × **5**
Square Matrix A matrix with the same number of rows as columns:2×2 , 3×3,4×4 are all square matrices. **Identity Matrix** Has **1** in each of the positions in the main diagonal and **0** elsewhere. **Note that** : **I** is a Square matrix.	*Example2:* **I =** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the **2** × **2** Identity matrix **I =** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the **3** × **3** Identity matrix
Matrix Addition If **A** and **B** are two matrices of the same size then we define **A+B** to be the matrix whose elements are the sums of the corresponding elements in **A** and **B**. **Only matrices of the same size can be added.** **A + (B + C) = (A+B)+C** **A – B = A + (-B)** **k(A+B) = kA + kB**	*Example3:* $\begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 7 \\ -9 & 1 & -6 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 4 & 1 \\ 1 & 5 & 4 \\ 4 & 5 & 1 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 9 & 0 \\ 1 & 8 & 11 \\ -5 & 6 & -5 \\ 4 & 4 & 10 \end{pmatrix}$

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Matrix Multiplication

For the product of two matrices **A** and **B** to be defined, the number of columns of **A** must be the same as the number of rows in **B**:

A : m×**n** ; B : **n**×p then **AB** is defined and of rank **m×p**

Properties

For any matrices **A , B , C** such that all the indicated sums and products exist:

A(BC) = (AB)C

A(B+C) = AB + BC

Remark

In general, AB may not equal BA.

Example5: suppose **A** is 2x**3** and **B** is **3**x5 then **AB** is defined, but **BA** is not defined. B(3x**5**) , A(**2**x3)

Remark

For any matrix **A** such that all the indicated products exist:

IA = AI = A

Where **I** is the identity matrix.

Determinant of a square matrix

To every **square** matrix **A**, there is an assigned number called the determinant of **A.** Written **det A** or **|A|**.

Determinant of a 2x2 matrix

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

|A| = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ ad – bc

Determinant of a 3x3 matrix

A= $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **then**

$$\mathbf{A^{-1}} = \mathbf{a}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \mathbf{b}\begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathbf{c}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example4:

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 3 & 5 & 1 \\ 6 & 7 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 20 & 26 & 17 \\ 38 & 49 & 41 & 24 \end{pmatrix}$$

The product is obtained by multiplying each row of **A** by the columns of **B**(first by first and so on)

The first entry:

$$(2 \quad 3 \quad 1)\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = 2x1+ 3x2 +1x6 = 14$$

The second entry:

$$(2 \quad 3 \quad 1)\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = 2x2+ 3x3 +1x7 = 20$$

and so on ……

Example6:

1. **A=** $\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$; **|A|**= $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ =10-(-3)=13

2. **A=** $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$

$$\mathbf{|A| = 2}\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} \mathbf{- 3}\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} \mathbf{+(-4)}\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}$$

|A| = 2(-20 + 2) – 3(0 – 2) -4(0 +4)

|A| = 2(-18) -3(-2) -4(4)

|A| = -36 + 6 – 16 = - 46

Inverse Matrix

A **square** matrix **A** has an inverse $\mathbf{A^{-1}}$ if $\mathbf{AA^{-1} = A^{-1}A = I}$

Remarks

1.Only square matrices may admit an inverse.

2.When a square matrix has an inverse,it has only one(unique)

3.A square matrix may have **no** inverse.**If |A| = 0 then $\mathbf{A^{-1}}$ does not exist.**

Inverse of a 2x2 matrix

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$\mathbf{A^{-1}} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Inverse of a 3x3 matrix

A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

We define the *cofactor* of a_{ij} denoted by A_{ij} as :

$\mathbf{A_{ij} = (-1)^{i+j}\,|M_{ij}|}$

We call the determinant $\mathbf{|M_{ij}|}$, the *minor* of a_{ij} .

The above matrix has 9 cofactors :

$\mathbf{A_{11}}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$\mathbf{A_{12}}=(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

and so on ……

$\mathbf{A^{-1}} = \dfrac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

Note the way in which $\mathbf{A_{ij}}$'s are placed.

Example7:

Note that ,for a 2x2 matrix, the inverse is obtained by switching the "diagonal" terms a and d, changing the sign of the "off diagonal" terms b and c and finally dividing by the determinant of the matrix: ad – bc

If ad – bc = 0 ,then $\mathbf{A^{-1}}$ does not exist.

A= $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$; ad –bc = (1)(4)–(3)(-2)=10

$\mathbf{A^{-1}} = \dfrac{1}{10}\begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{10} & \frac{2}{10} \\ \frac{-3}{10} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{-3}{10} & \frac{1}{10} \end{pmatrix}$

The matrix **A =** $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ has no inverse

since**|A|** = ad – bc = (2)(3) – (1)(6) = 0

*Example8:***A=** $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ we know

|A| = -46 from Example6 above.

$\mathbf{A_{11}}=(-1)^{1+1}\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18$

$\mathbf{A_{12}}=(-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = -(-2) = +2$

$\mathbf{A_{13}}=(-1)^{1+3}\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$

Similarly: $\mathbf{A_{21}= -11, A_{22}=14, A_{23}=5, A_{31}=-10, A_{32}=-4, A_{33}=-8}$ **; then $\mathbf{A^{-1}}$ =**

$$\frac{1}{-46}\begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{pmatrix} = \begin{pmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ \frac{-1}{23} & \frac{-7}{23} & \frac{2}{23} \\ \frac{-2}{23} & \frac{-5}{46} & \frac{4}{23} \end{pmatrix}$$

Multiplying each entry by -1/46 .

Solving Systems with matrices

Consider the system:

ax + by = c
a′x + b′y = c′

Let **A** = $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$; **X** = $\begin{pmatrix} x \\ y \end{pmatrix}$

and **B**= $\begin{pmatrix} c \\ c' \end{pmatrix}$

Since

AX = $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ a'x+b'y \end{pmatrix}$

the original system is equivalent to the single matrix system : **AX = B**

$\mathbf{A^{-1}\,AX = A^{-1}\,B}$ (multiplying both sides by $\mathbf{A^{-1}}$)

$\mathbf{IX = A^{-1}B}$ ($\mathbf{A^{-1}A = I}$)

$\mathbf{X = A^{-1}B}$ (**IX = X**)

Conclusion

If **AX = B** ⇔ $\mathbf{X = A^{-1}B}$

That is, **AX = B** has a solution if and only if $\mathbf{A^{-1}}$ exists

This implies the following:

A square matrix **A** is invertible(has an inverse) if and only if **AX = B** has a unique solution.

Example9: Solve the system :

x + 1.5y = 8
2x + 3y = 10

A= $\begin{pmatrix} 1 & 1.5 \\ 2 & 3 \end{pmatrix}$, since the determinant of **A:**

|A| = (1)(3) – (2)(1.5) = 0

then $\mathbf{A^{-1}}$ **does not exist** and hence the above system has **no solution**.

Example10:

Solve the system :

-x -2y + 2z = 9
2x + y –z = - 3
3x – 2y + z = -6

We have **AX = B** where

A = $\begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix}$; **X** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$; **B** = $\begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix}$

Then $\mathbf{X = A^{-1}B}$

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{5}{3} & \frac{7}{3} & -1 \\ \frac{7}{3} & \frac{8}{3} & -1 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{5}{3} & \frac{7}{3} & -1 \\ \frac{7}{3} & \frac{8}{3} & -1 \end{pmatrix}\begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 14 \\ 19 \end{pmatrix}$$

Thus the solution is (1, 14, 19)

Example11: Solve the matrix system

X = D + AX

X – AX = D with **X = IX**

IX –AX = D

(I – A)X = D

$\mathbf{X = (I - A)^{-1}\,D}$

Remember this result , we'll use it frequently in matrix applications specially input – output analysis.