International Institute
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Unit 76: Management Mathematics

Handout #6b

Differential Equations II

Topic

Linear Equations

$\frac{dy}{dt}$ + Py = Q

Where P & Q are functions of t only.

Solution :

y= $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{d}t + c\right)$

<u>Example 2</u>: $\frac{dy}{dt}$ + 3y = 4

y= $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{d}t + c\right)$

y= $e^{\int -3dt}\left(\int 4e^{\int 3dt}\,\mathrm{d}t + c\right)$

y = e^{-3t} ($\int 4e^{3t}dt$ + c)

y = e^{-3t} (4(1/3)e^{3t} + c)

Second order Equation:

$\frac{d^2y}{dt^2}$ + a$\frac{dy}{dt}$ + by = f(t)

Similar situation to *Difference Equations.*

The general solution :

y = y_c + y_p

where y_c is the complementary function and y_p is the particular integral.

Auxiliary Equation:

$$r^2 + ar + b = 0$$

Case 1 : r_1 and r_2 are real distinct.

y_c = $Ae^{r_1t} + Be^{r_2t}$

Case 2 : r_1, r_2 are real and equal; r = r_1= r_2

yc = $(A + Bt)e^{rt}$

Interpretation

<u>Example1</u>:

$\frac{dy}{dx} - 2y = e^x$

P = -2 ; Q = e^x

y= $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{d}t + c\right)$

y= $e^{\int 2xdx}\left(\int e^x e^{\int -2xdx}\,\mathrm{d}x + c\right)$

y = e^{2x}($\int e^x(\frac{-1}{2})e^{-2x}dx + c)$

y = e^{2x} (-½ $\int e^{-x}dx + c$)

y = e^{2x} (½ e^{-x} + c)

y = ½ e^x + ce^{2x}

<u>Example 3</u>: $\frac{d^2y}{dt^2}$ - 3$\frac{dy}{dt}$ + 2y = te^{3t}

Auxiliary Equation: $r^2 - 3r + 2 = 0$

$\Rightarrow$ r = 1 ; r = 2

y_c = $Ae^t + Be^{2t}$

y_p = (C +Dt)e^{3t} to substitute this in the equation, we need y_p' and y_p''

y_p' =De^{3t} + 3(C+Dt)e^{3t} = (3C+D+3Dt)e^{3t}

y_p'' = 3De^{3t} + 3(3C+D+3Dt)e^{3t}

= (9C + 6D + 9Dt)e^{3t}

Substituting in the equation:

(9C + 6D+ 9Dt)e^{3t} -3(3C + D+ 3Dt)e^{3t} + 2(C +Dt)e^{3t} = te^{3t}

(2C + 3D +Dt)e^{3t} = te^{3t}

D = 1 ; 2C + 3D = 0 ; C = -3/2

General solution : **y = y_c + y_p**

y = $Ae^t + Be^{2t}$ + ($\frac{-3}{2}$ + t)e^{3t}

Case 3 : r_1, r_2 are imaginary

$y_c = e^{\frac{-a}{2}t}(A\cos\alpha\, t + B\sin\alpha\, t)$

where $\alpha = \dfrac{\sqrt{4b-a^2}}{2}$

PS : Because the solution depends on a and b; if the equation is given in the form :
2y″ + 6y′ +4y= 0
You need to divide by 2 to get the correct values of a and b :
y″ + 3y′+ 2y = 0

Finding the Particular Solution :

f(t)	**y_p**
t^n e.g. t + 3 $2t^2$ 5	$A_0+A_1t+\dots+A_nt^n$ C + Dt $C + Dt + Et^2$ C+Dt
a^t e.g. 2^t $2^t + t$	Ca^t $C2^t$ $C2^t + Dt + E$
$a^t\, t^n$ e.g. $2^t\,(t)$	$a^t(A_0+A_1t+\dots+A_nt^n)$ $2^t(C + Dt)$
a^t sinbt e.g. sin2t	a^t(Acosbt + Bsinbt) Ccos2t + Dsin2t
a^t cosbt e.g. $\cos\pi t$	a^t(Acosbt + Bsinbt) $C\cos\pi t + D\sin\pi t$

Examples :
f(t) = 3t + 2 ; y_p= C + Dt
f(t) = t^3 ; y_p= $C + Dt + Et^2 + Ft^3$
f(t) = 2cos3t + t ;
y_p= Ccos3t + D sin3t + E + Ft;
f(t) = e^{2t} (3- 5t) ; y_p= e^{2t} (C+Dt)
f(t) = 3 + sin2t + e^{-3t}
y_p=C+Dt + Ecos2t+ Fcos2t+ Ge^{-3t}

f(t) = t^2sin 6t + (2t – 1)cos 6t
take the highest polynomial degree
$y_p=(C+Dt+Et^2)\sin 6t + (F+Gt+Ht^2)\cos 6t$

Example 4:

$\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$

Auxiliary Equation: $r^2 + 6r + 9 = 0$
⇒ r = -3; r= -3 ; tow equal roots.
$y_c = (A + Bx)e^{-3x}$
There is no need for a particular integral as f(x)= 0 .

Example 5:

$\dfrac{d^2y}{dx^2} + y = e^x$; y(0) = 1 ; y′(0) = 11/2

Auxiliary Equation: $r^2 + 1 = 0$; r = $\pm i$
Complex roots.

$y_c = e^{\frac{-a}{2}x}(A\cos\alpha\, x + B\sin\alpha\, x)$

$\alpha = \dfrac{\sqrt{4b-a^2}}{2} = \dfrac{\sqrt{4(1)-0^2}}{2} = 1$

$y_c = e^{\frac{-0}{2}x}(A\cos x + B\sin x)$
$y_c = A\cos x + B\sin x$

$y_p = Ce^x$; $y_p' = Ce^x$; $y_p'' = Ce^x$

$\dfrac{d^2y}{dx^2} + y = e^x \Rightarrow Ce^x + Ce^x = e^x$

$2Ce^x = e^x \Rightarrow C = ½ \Rightarrow y_p = ½ e^x$
General solution : $\mathbf{y = y_c + y_p}$
y= $A\cos x + B\sin x$ + ½ e^x

y(0) = 1 ⇒ Acos0 + Bsin0 + ½e^0 = 1

⇒ A + ½ = 1 ⇒ A = ½

y′(0) =11/2 ⇒-Asin0+ Bcos0+½e^0= 11/2
⇒ B+ ½ =11/2 ⇒ B = 5

y= $\dfrac{1}{2}\cos x + 5\sin x$ + ½ e^x

About Particular solutions
In some cases you may substitute the particular solution in the equation and get no values for the constants.This occurs when the **particular solution is part of the complementary function.**

$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^{4x}$

$r^2 - 5r + 4 = 0 \Rightarrow r = 1 ; r = 4$

$y_c = Ae^{x} + Be^{4x}$

Note: e^{4x} is part of y_c
The particular solution $y_p = Ce^{4x}$ will not work!

$y_p = Ce^{4x}; y_p' = 4Ce^{4x}; y_p'' = 16Ce^{4x}$

Substituting in the equation

$16Ce^{4x} - 20Ce^{4x} + 4Ce^{4x} = e^{4x}$

$\Rightarrow (0)e^{4x} = e^{4x}$??

To fix it we attach x to Ce^{4x}:

Let $y_p = Cxe^{4x}$; $y_p' = (4Cx+C)e^{4x}$; $y_p'' = (16Cx+8C)e^{4x}$

$(16Cx+8C)e^{4x} - 5(4Cx+C)e^{4x} + 4Cxe^{4x} = e^{4x}$

$3Ce^{4x} = e^{4x} \Rightarrow C = 1/3$

$\Rightarrow y_p = (1/3)xe^{4x}$ **(see examples 6)**

Conditions for oscillation:
assume the roots of the auxiliary equation are r_1 and r_2;time path is oscillating if both roots are complex.

Conditions for convergence:

1. Two real distinct roots:
Both *negative* :$r_1 < 0$ and $r_2 < 0$ **Converges.**
If one of the roots is *positive*; **Diverges.**

2. Two real equal roots :
If the repeated root is *negative*; **Converges.**
If the repeated root is *positive*; **Diverges.**

3. Complex roots :

$e^{kt}(A\cos\alpha t + B\sin\alpha t)$

If $K < 0$; **converges.**

As t--→∞ , e^{-t} -----→ 0

As t--→∞ , e^{t}--→∞ **(see examples7)**

Examples 6:

1.) $y'' + 3y' - 10y = 3t + e^{-5t} - 1$

The auxiliary roots : r = 2 , r = -5

$y_c = Ae^{2t} + Be^{-5t}$

The original particular solution:

$y_p = C + Dt + Ee^{-5t}$ **will not work! Since** e^{-5t} is part of the complimentary function.

To fix it we attach t to Ee^{-5t}: the correct one : $y_p = C + Dt + Ete^{-5t}$

2.) $y'' - 9y = 5t^2e^{3t} + t\cos t - \sin t$

$y_c = Ae^{-3t} + Be^{3t}$

The original particular solution:

$y_p = (C+Dt+Et^2)e^{3t} + (Ft+G)\cos t + (H+It)\sin t$ **will not work! Since** e^{3t} is part of the complimentary function.

To fix it we attach t to $(C + Dt + Et^2)e^{3t}$**:** the correct one :

$y_p = t(C+Dt+Et^2)e^{3t} + (Ft+G)\cos t + (H+It)\sin t$

3.) $y'' + 4y' + 4y = (2-3t^2)e^{-2t}$

$y_p = t^2(C+Dt+Et^2)e^{-2t}$ **why?**

Examples 7:

1. $y = 5e^{-t} - 3e^{-2t} + 7$
Both roots are negative;converges.
As t-----→∞ **; e^{-t} -→ 0 ; e^{-2t} -→ 0**
y converges to 7 .

2. $y = -2e^{3t} - e^{-2t} + 2$
One of the roots : 3 > 0 ;diverges.
As t---→∞ **; e^{3t} -→∞ ; e^{-2t} -→ 0**

3. $y = e^{5t} - 3e^{2t} + t + 1$
Both roots are positive ;diverges.
As t--->∞ **; e^{5t} -→∞; e^{2t}-→∞**

4. $y = (2 + 3t)e^{4t}$;one positive repeated root ; diverges.

5. $y = (2 - t)e^{-7t}$**;** one negative repeated root; converges.

PS: if both roots are negative or the repeated root is negative and there is a particular integral y_p then the behavior depends on y_p on the long run.

6. $y = 5e^{-t} - 3e^{-2t} + t + 1$
As t-----→∞ **; e^{-t} -→ 0 ; e^{-2t} -→ 0**
On the long run it depends on t + 1.

7. $y = e^{-3t}(2\cos 5t + 4\sin 5t)$
-3 < 0 ; it converges.

Economic Applications

1. Difference vs. Differential

Difference equations are related to *dynamics* economy using ***discrete-time*** models.This means the time periods involved are ***successive time periods of equal length*** (calendar years for instance) and the various quantities are measured over those periods.

For example I_t denotes the total investment over the t^{th} calendar year and the behavior of the sequences I_0 , I_1 , I_2,…. are to be considered.

This is the reason(discrete time) behind graphing the solution of a difference equation as a *step function.*

Differential Equations are related to *dynamics* economy using ***continuous-time*** models.

I(t) denotes the total investment; t is a continuous varying parameter,for example one values of t could be 12:00 noon on a given day.

2. Market trends and Consumer's demand

In certain markets,consumer's often try to anticipate trends. Their demand is not simply based on the selling price but how fast this price has been rising or falling.for example if the price of houses is falling fast ,they decide to wait,as a result the demand for property is reduced.

Application of difference equations:

I_t : Investment ; Y_t : Income

C_t :Consumption ; Q_t :Production

The equilibrium conditions:

$Q_t = Y_t$; $Y_t = C_t + I_t$

A behavioral condition is :

This years consumption is linearly related to last year's income.

$C_t = a + bY_{t-1}$; a , b > 0

Another behavioral condition is :

This year investment is linearly related to last year's increase in production:

$I_t = c + w(Q_{t-1} - Q_{t-2})$; c ,w > 0

Using the equilibrium conditions :

$Q_t = Y_t$; $Y_t = C_t + I_t$

The second order difference equation:

$Y_t - (b+w)Y_{t-1} + wY_{t-2} = a + c$

Is obtained.

Application of differential equations:

I(t) : Investment (note I(t) instead of I_t)

Y(t) :Income ; C(t) : Consumption

Q(t) : Production.

Equilibrium conditions:

Q(t) = Y(t) and Y(t) = C(t) + I(t)

Suppose the consumption C(t) and the income Y(t) are linked by the equation:

C(t) = a + b Y(t) ; a,b constants.Then:

$\frac{dC}{dt} = b\frac{dY}{dt}$ If C is a function of Y :

$\frac{dC}{dY} = b$; b is called the marginal propensity to consume.

<u>Example 8:</u>Suppose consumer's demand:

q = $9 - 6p + 5\frac{dp}{dt} - 2\frac{d^2p}{dt^2}$; the supply :

q = $-3 + 4p - \frac{dp}{dt} - \frac{d^2p}{dt^2}$; at equilibrium :

supply = demand ;the second order DE:

$$\frac{d^2p}{dt^2} - 7\frac{dp}{dt} + 6p = 12$$