

International Institute for Technology and Management



October 25th, 2005

Unit 76: Management Mathematics

Handout #5c

Difference Equations III

Topic	Interpretation
<p>Solution of Difference Equations (2) Linear Second Order:</p> <p>a. Homogenous equation: $y_{t+2} + ay_{t+1} + by_t = 0$ where a, b are constants and $b \neq 0$; $t = 0, 1, 2, \dots$ The general solution : $y_t = y_c$ where y_c is the complementary function $y_c = Ar_1^t + Br_2^t$; A, B are some constants. r_1 and r_2 are the roots of the <i>auxiliary</i> equation : $r^2 + ar + b = 0$</p> <p>Case 1 : r_1 and r_2 are real distinct. $y_t = Ar_1^t + Br_2^t$</p> <p>Case 2 : r_1, r_2 are real and equal; $r = r_1 = r_2$ $y_t = (A + Bt)r^t$</p> <p>Case 3 : r_1, r_2 are imaginary $r_1 = g + ih, r_2 = g - ih$ $g = \frac{-a}{2}$; $h = \frac{1}{2}\sqrt{4b - a^2}$ $y_t = (\sqrt{b})^t (A \cos \alpha t + B \sin \alpha t)$ A, B constants ; $\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right)$</p>	<p>Examples:</p> <p>(1) $y_{t+2} - 2y_{t+1} - 15y_t = 0$ Given $y_0 = 3$; $y_1 = 7$ the <i>auxiliary</i> equation : $r^2 - 2r - 15 = 0$ $(r + 3)(r - 5) = 0 \Rightarrow r = -3$; $r = 5$ two distinct real roots: $y_t = A(-3)^t + B(5)^t$ $t=0$: $y_0 = 3 = A(-3)^0 + B(5)^0 = A + B$ $t=1$: $y_1 = 7 = A(-3)^1 + B(5)^1 = -3A + 5B$ Solving $-3A + 5B = 7$; $A + B = 3$ simultaneously : $A = 1, B = 2$ $\Rightarrow y_t = A(-3)^t + B(5)^t = (1)(-3)^t + 2(5)^t$</p> <p>(2) $y_{t+2} - 6y_{t+1} + 9y_t = 0$ the <i>auxiliary</i> equation : $r^2 - 6r + 9 = 0$ $(r - 3)^2 = 0 \Rightarrow r = 3$; $r = 3$ two equal real roots: $y_t = (A + Bt)(3)^t$</p> <p>(3) $y_{t+2} + 0.5y_{t+1} + 0.25y_t = 0$ the <i>auxiliary</i> equation: $r^2 + 0.5r + 0.25 = 0$ $r = \frac{-0.5 \pm \sqrt{-0.75}}{2} = -0.25 \pm \frac{i}{2}\sqrt{0.75}$ $y_t = (\sqrt{0.25})^t (A \cos \alpha t + B \sin \alpha t)$ $\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}\left(-0.5/2\sqrt{0.25}\right)$ $= \cos^{-1}(-0.5) = \frac{2\pi}{3}$ $y_t = (0.5)^t (A \cos \frac{2\pi}{3} t + B \sin \frac{2\pi}{3} t)$</p>

b. Non Homogenous Equation:

$$y_{t+2} + ay_{t+1} + by_t = r_t$$

where a, b are constants and $b \neq 0$; $t = 0, 1, 2, \dots$

The general solution :

$$y_t = y_c + y_p$$

where y_c is the complementary function and y_p is the particular solution.

$y_c = Ar_1^t + Br_2^t$; A, B are some constants.

r_1 and r_2 are the roots of the auxiliary equation :

$$r^2 + ar + b = 0$$

Finding the Particular Solution

Case 1 : $r_t = c$ is constant

$$y_{t+2} + ay_{t+1} + by_t = c$$

The particular solution is :

$$y_p = \frac{c}{1+a+b}; a+b \neq -1$$

$$y_p = \frac{c}{2+a}t; a+b = -1; a \neq -2$$

$$y_p = \frac{c}{2}t^2; a+b = -1; a = -2$$

Conditions for oscillation: assume the roots of the auxiliary equation are r_1 and r_2 ; time path is oscillating in the following cases:

1. The dominant root is negative: $r_1 < 0$; $|r_1| > r_2$
2. Both roots are negative.
3. Both roots are complex.

Conditions for convergence:

1. The dominant root > 1 : $|r_1| > 1$ Diverges.

2. The dominant root < 1 : $|r_1| < 1$ Converges.

Example 2: Find

(a) the particular solution
(b) the complementary function (c) the general solution

(d) the definite solution (e) comment on the dynamic stability of :

1. $y_{t+2} + 7y_{t+1} + 6y_t = 42$;

$$y_0 = 16, y_1 = -35$$

2. $y_{t+2} + 2y_t = 24$;

$$y_0 = 11, y_1 = 18$$

3. $y_{t+2} - 11y_{t+1} + 10y_t = 27$

$$y_0 = 2, y_1 = 53$$

Example 2 Answers:

1. $y_{t+2} + 7y_{t+1} + 6y_t = 42$

(a) $a+b = 7 + 6 = 13 \neq -1$

$$y_p = \frac{c}{1+a+b} = 3$$

(b) The auxiliary roots : $r^2 + 7r + 6 = 0$
 $r = -1, r = -6$

The complementary function:

$$y_c = Ar_1^t + Br_2^t = A(-1)^t + B(-6)^t$$

(c) The general solution: $y_t = y_c + y_p$
 $y_t = A(-1)^t + B(-6)^t + 3$

(d) $y_0 = 16 \Rightarrow A+B+3 = 16$

$$y_1 = -35 \Rightarrow -A - 6B + 3 = -35$$

Solving simultaneously $A = 8, B = 5$

The definite solution:

$$y_t = 8(-1)^t + 5(-6)^t + 3$$

(e) The auxiliary root with the largest absolute value is called the *dominant* root because it dominates the time path.

In this case -6 is the dominant root since

$$|-6| = 6 > |-1| = 1$$

$-6 < 0$ it **oscillates**; since $|-6| > 1$ it

diverges (explodes). Therefore the time path is divergent oscillating.

2. $y_{t+2} + 2y_t = 24$

(a) $a+b = 0 + 2 \neq -1$;

$$y_p = \frac{c}{1+a+b} = 8$$

Case 2 : r_t is NOT constant

$y_{t+2} + ay_{t+1} + by_t = r_t$
The following table gives the trial y_p for different values of r_t

r_t	y_p
t^n e.g. $t + 3$ $2t^2$	$A_0 + A_1t + \dots + A_nt^n$ $A + Bt$ $A + Bt + Ct^2$
a^t e.g. 2^t $2^t + t$	Aa^t $A2^t$ $A2^t + Bt + C$
$a^t t^n$ e.g. $2^t (t)$	$a^t(A_0 + A_1t + \dots + A_nt^n)$ $2^t(Bt + C)$
$a^t \sin bt$ e.g. $\sin 2t$	$a^t(A \cos bt + B \sin bt)$ $A \cos 2t + B \sin 2t$
$a^t \cos bt$ e.g. $\cos \pi t$	$a^t(A \cos bt + B \sin bt)$ $A \cos \pi t + B \sin \pi t$

Solution method: Substitute the particular solution in the original equation to find the constants.

Examples :

$r_t = 3t + 2 ; y_p = A + Bt$

$r_t = t^3 ; y_p = A + Bt + Ct^2 + Dt^3$

$r_t = 2\cos 3t + t ;$

$y_p = A \cos 3t + B \sin 3t + C + Dt$

Examples: Solve the following equations:

1. $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$
 $y_1 = 1, y_2 = 3$

2. $y_{t+2} - 5y_{t+1} + 6y_t = 3^t + t$

3. $2y_{t+2} - y_{t+1} - y_t = \sin \pi t$

(b) The auxiliary roots : $r^2 + 2 = 0$

$r = -i\sqrt{2}, r = +i\sqrt{2}$

The complementary function:

$y_c = (\sqrt{2})^t (A \cos \alpha t + B \sin \alpha t)$

$\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

$y_t = (\sqrt{2})^t \left(A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t\right)$

(c) The general solution: $y_t = y_c + y_p$

$y_t = (\sqrt{2})^t \left(A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t\right) + 8$

(d) $y_0 = 11 \Rightarrow A + 8 = 11 \Rightarrow A = 3$

$y_1 = 18 \Rightarrow \sqrt{2}(0 + B) + 8 = 18$

$\Rightarrow B = \frac{10}{\sqrt{2}} = 7.07$

$y_t = (\sqrt{2})^t \left(3 \cos \frac{\pi}{2} t + 7.07 \sin \frac{\pi}{2} t\right) + 8$

(e) with $\sqrt{2} > 1$, the time path is divergent. Oscillating for having complex auxiliary roots. Divergent oscillating.

3. $y_{t+2} - 11y_{t+1} + 10y_t = 27$

(a) $a + b = -11 + 10 = -1 ; b \neq -2$

$y_p = \frac{c}{2+a} t = -3t$

(b) The auxiliary roots : $r^2 - 11r + 10 = 0$
 $r = 1, r = 10$

The complementary function:

$y_c = Ar_1^t + Br_2^t = A(1)^t + B(10)^t = A + B(10)^t$

(c) The general solution: $y_t = y_c + y_p$

$y_t = A + B(10)^t - 3t$

(d) $y_0 = 2 \Rightarrow A + B = 2$

$y_1 = 53 \Rightarrow A + 10B - 3 = 53$

Solving simultaneously $A = -4, B = 6$

The definite solution:

$y_t = -4 + 6(10)^t - 3t$

(e) The time path is divergent since the dominant root 10 is greater than 1.

Non-oscillating since 10 is positive.

1. $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$
 $y_1 = 1, y_2 = 3$

The auxiliary equation : $9r^2 + 6r + 1 = 0$ i.e. $(3r + 1)(3r + 1) = 0$
 $r = -1/3$ two equal real roots.

The complementary function : $y_c = (A + Bt)(-1/3)^t$

For a particular solution ,try $y_p = C + Dt$ substitute this in the original

Equation : $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$

$$\Rightarrow 9[C + D(t+2)] + 6[C + D(t+1)] + C + Dt = 2t + 1$$

$$\Rightarrow (9D + 6D + D)t + 9C + 6C + C + 18D + 6D = 2t + 1$$

$\Rightarrow 16Dt + 16C + 24D = 2t + 1$ equating coefficients of t and the constant terms:
 $16D = 2 ; D = 1/8 ; 16C + 24(1/8) = 1 ; C = -2/16 = -1/8$

Hence $y_p = (1/8)t - 1/8$

The general solution : $y_t = y_c + y_p = (A + Bt)(-1/3)^t + (1/8)t - 1/8$

Now use $y_1 = 1, y_2 = 3$ to find A and B

2. $y_{t+2} - 5y_{t+1} + 6y_t = 4^t + t$

The auxiliary equation : $r^2 - 5r + 6 = 0$ i.e. $(r - 2)(r - 3) = 0 ; r = 2, r = 3$.

The complementary function : $y_c = A(2)^t + B(3)^t$

For a particular solution ,try $y_p = C4^t + D + Et$ substitute this in the original Equation : $y_{t+2} - 5y_{t+1} + 6y_t = 4^t + t$

$$C4^{t+2} + D + E(t+2) - 5[C4^{t+1} + D + E(t+1)] + 6(C4^t + D + Et) = 4^t + t$$

to find $C = 1/2, D = 3/4$ and $E = 1/2$:

$$y_p = C4^t + D + Et = (1/2) 4^t + 3/4 + t/2$$

$$y_t = y_c + y_p = A(2)^t + B(3)^t + (1/2) 4^t + 3/4 + t/2$$

3. $2y_{t+2} - y_{t+1} - y_t = \sin \pi t$

The auxiliary equation : $2r^2 - r - 1 = 0$ i.e. $(2r - 1)(r + 1) = 0$

$r = 1/2, r = -1$.

The complementary function : $y_c = A(1/2)^t + B(-1)^t = A(1/2)^t + B(-1)^t$

For a particular solution ,try $y_p = C \cos \pi t + D \sin \pi t$ substitute this in the original Equation : $2y_{t+2} - y_{t+1} - y_t = \sin \pi t$ to find C and D.

$$2(C \cos(t+2)\pi + D \sin(t+2)\pi) - (C \cos(t+1)\pi + D \sin(t+1)\pi) - (C \cos \pi t + D \sin \pi t) = \sin \pi t$$

Now $\sin(t+2)\pi = \sin \pi t, \cos(t+2)\pi = \cos \pi t, \sin(t+1)\pi = -\sin \pi t, \cos(t+1)\pi = -\cos \pi t$:

$$2(C \cos \pi t + D \sin \pi t) - (-C \cos \pi t - D \sin \pi t) - (C \cos \pi t + D \sin \pi t) = \sin \pi t$$
$$2C \cos \pi t - 2D \sin \pi t = \sin \pi t ; C = 0 ; D = -1/2$$

$$y_t = y_c + y_p = A(1/2)^t + B(-1)^t - 1/2 \sin \pi t$$