International Institute
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Unit 76: Management Mathematics

Handout #5c

Difference Equations III

Topic	Interpretation
Solution of Difference Equations **(2) Linear Second Order:** **a. Homogenous equation:** $y_{t+2} + ay_{t+1} + by_t = 0$ where a , b are constants and $b \neq 0$; $t = 0,1,2,\ldots$ The general solution : $y_t = y_c$ where y_c is the complementary function $y_c = Ar_1^t + Br_2^t$; A , B are some constants. r_1 and r_2 are the roots of the *auxiliary* equation : $r^2 + ar + b = 0$ **Case 1 :** r_1 and r_2 are real distinct. $y_t = Ar_1^t + Br_2^t$ **Case 2 :** r_1, r_2 are real and equal; $r = r_1 = r_2$ $y_t = (A + Bt)r^t$ **Case 3 :** r_1, r_2 are imaginary $r_1 = g + ih$, $r_2 = g - ih$ $g = \frac{-a}{2}$; $h = \frac{1}{2}\sqrt{4b - a^2}$ $y_t = (\sqrt{b})^t(A\cos\alpha\, t + B\sin\alpha\, t)$ A,B constants ; $\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}})$	<u>Examples</u>: (1) $y_{t+2} - 2y_{t+1} - 15y_t = 0$ Given $y_0 = 3$; $y_1 = 7$ the *auxiliary* equation : $r^2 - 2r - 15 = 0$ $(r+3)(r-5) = 0 \Rightarrow r = -3$; $r = 5$ two distinct real roots: $y_t = A(-3)^t + B(5)^t$ $t=0$: $y_0 = 3 = A(-3)^0 + B(5)^0 = A + B$ $t=1$: $y_1 = 7 = A(-3)^1 + B(5)^1 = -3A + 5B$ Solving $-3A + 5B = 7$; $A + B = 3$ simultaneously : $A = 1$, $B = 2$ $\Rightarrow y_t = A(-3)^t + B(5)^t = (1)(-3)^t + 2(5)^t$ (2) $y_{t+2} - 6y_{t+1} + 9y_t = 0$ the *auxiliary* equation : $r^2 - 6r + 9 = 0$ $(r-3)^2 = 0 \Rightarrow r = 3$; $r = 3$ two equal real roots: $y_t = (A + Bt)(3)^t$ (3) $y_{t+2} + 0.5y_{t+1} + 0.25y_t = 0$ the *auxiliary* equation: $r^2 + 0.5r + 0.25 = 0$ $r = \frac{-0.5 \pm \sqrt{-0.75}}{2} = -0.25 \pm \frac{i}{2}\sqrt{0.75}$ $y_t = (\sqrt{0.25})^t(A\cos\alpha\, t + B\sin\alpha\, t)$ $\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(-0.5/2\sqrt{0.25})$ $= \cos^{-1}(-0.5) = \frac{2\pi}{3}$ $y_t = (0.5)^t(A\cos\frac{2\pi}{3}t + B\sin\frac{2\pi}{3}t)$

b. Non Homogenous Equation:

$y_{t+2} + ay_{t+1} + by_t = r_t$
where a , b are constants and
$b \neq 0$; t = 0,1,2,…..

The general solution :

$$\mathbf{y_t = y_c + y_p}$$

where y_c is the complementary function and y_p is the particular solution.

$y_c = Ar_1^t + Br_2^t$; A , B are some constants.

r_1 and r_2 are the roots of the *auxiliary* equation :

$$r^2 + ar + b = 0$$

Finding the Particular Solution

Case 1 : r_t = c is constant

$y_{t+2} + ay_{t+1} + by_t = c$

The particular solution is :

$$y_p = \frac{c}{1+a+b} \; ; a+b \neq -1$$

$$y_p = \frac{c}{2+a} t \; ; a+b = -1 ; \; a \neq -2$$

$$y_p = \frac{c}{2} t^2 \; ; \; a+b = -1 \; ; \; a = -2$$

Conditions for oscillation: assume the roots of the auxiliary equation are r_1 and r_2;time path is oscillating in the following cases:

1. The dominant root is negative:**$r_1 < 0$; $|r_1| > r_2$**
2. Both roots are negative.
3. Both roots are complex.

Conditions for convergence:

1.The dominant root > 1 : $|r_1| > 1$
Diverges.

2.The dominant root < 1 : $|r_1| < 1$
Converges.

Example 2:Find
(a)the particular solution
(b)the complementary function (c) the general solution
(d)the definite solution (e)comment on the dynamic stability of :

1. $y_{t+2} + 7y_{t+1} + 6y_t = 42$; $y_0=16$, $y_1=-35$
2. $y_{t+2} + 2y_t = 24$; $y_0=11$, $y_1=18$
3. $y_{t+2} - 11y_{t+1} + 10y_t = 27$; $y_0=2$, $y_1=53$

Example 2 Answers:

1. $y_{t+2} + 7y_{t+1} + 6y_t = 42$

(a) a+b= 7 + 6 = 13 $\neq$ -1

$$y_p = \frac{c}{1+a+b} = 3$$

(b) The auxiliary roots : $r^2 + 7r + 6 = 0$
r = -1 , r = -6

The complementary function:

$y_c = Ar_1^t + Br_2^t = A(-1)^t + B(-6)^t$

(c) The general solution: $\mathbf{y_t = y_c + y_p}$

$y_t = A(-1)^t + B(-6)^t + 3$

(d) $y_0=16 \Rightarrow A+B+3 = 16$
$y_1=-35 \Rightarrow -A -6B+3 = -35$

Solving simultaneously A = 8 , B = 5

The definite solution:

$y_t = 8(-1)^t + 5(-6)^t + 3$

(e) The auxiliary root with the largest absolute value is called the *dominant* root because it dominates the time path.
In this case -6 is the dominant root since $|-6| = 6 > |-1| = 1$
-6 < 0 it **oscillates**; since |-6| > 1 it **diverges** (explodes).Therefore the time path is divergent oscillating.

2. $y_{t+2} + 2y_t = 24$

(a) a+b= 0+ 2 $\neq$ -1 ;

$$y_p = \frac{c}{1+a+b} = 8$$

Case 2 : r_t is NOT constant

$y_{t+2} + ay_{t+1} + by_t = r_t$

The following table gives the trial y_p for different values of r_t

r_t	y_p
t^n e.g. $t + 3$ $2t^2$	$A_0+A_1t+\ldots+A_nt^n$ $A + Bt$ $A + Bt + Ct^2$
a^t e.g. 2^t $2^t + t$	Aa^t $A2^t$ $A2^t + Bt + C$
$a^t t^n$ e.g. $2^t (t)$	$a^t(A_0+A_1t+\ldots+A_nt^n)$ $2^t(Bt + C)$
$a^t \sin bt$ e.g. $\sin 2t$	$a^t(A\cos bt + B\sin bt)$ $A\cos 2t + B\sin 2t$
$a^t \cos bt$ e.g. $\cos \pi t$	$a^t(A\cos bt + B\sin bt)$ $A\cos \pi t + B\sin \pi t$

***<u>Solution method</u>**: Substitute the particular solution in the original equation to find the constants.*

<u>Examples</u> :

$r_t = 3t + 2$; $y_p = A + Bt$

$r_t = t^3$; $y_p = A + Bt + Ct^2 + Dt^3$

$r_t = 2\cos 3t + t$;

$y_p = A\cos 3t + B \sin 3t + C + Dt$

<u>Examples</u>: Solve the following equations:

1. $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$
   $y_1=1$, $y_2= 3$
2. $y_{t+2} - 5y_{t+1} + 6y_t = 3^t + t$
3. $2y_{t+2} - y_{t+1} - y_t = \sin \pi t$

(b) The auxiliary roots : $r^2 + 2 = 0$

r = $-i\sqrt{2}$, r = $+i\sqrt{2}$

The complementary function:

$y_c = (\sqrt{2})^t (A\cos\alpha\, t + B\sin\alpha\, t)$

$\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(0) = \frac{\pi}{2}$

$y_t = (\sqrt{2})^t (A\cos\frac{\pi}{2} t + B\sin\frac{\pi}{2} t)$

(c) The general solution: $\mathbf{y_t = y_c + y_p}$

$y_t = (\sqrt{2})^t (A\cos\frac{\pi}{2} t + B\sin\frac{\pi}{2} t) + 8$

(d) $y_0=11 \Rightarrow A+8 = 11 \Rightarrow A = 3$

$y_1=18 \Rightarrow \sqrt{2}(0 + B)+8 = 18$

$\Rightarrow B = \frac{10}{\sqrt{2}} = 7.07$

$y_t = (\sqrt{2})^t (3\cos\frac{\pi}{2} t + 7.07\sin\frac{\pi}{2} t) + 8$

(e) with $\sqrt{2}$ **> 1** ,the time path is divergent. Oscillating for having complex auxiliary roots. Divergent oscillating.

3. $y_{t+2} - 11y_{t+1} + 10y_t = 27$

(a) $a + b = -11+10 = -1$; $b \neq -2$

$y_p = \frac{c}{2+a} t$ = -3t

(b) The auxiliary roots : $r^2 - 11r + 10 = 0$

r = 1 , r = 10

The complementary function:

$y_c = Ar_1^t + Br_2^t = A(1)^t + B(10)^t = A + B(10)^t$

(c) The general solution: $\mathbf{y_t = y_c + y_p}$

$y_t = A + B(10)^t - 3t$

(d) $y_0=2 \Rightarrow A + B = 2$

$y_1=53 \Rightarrow A + 10B - 3 = 53$

Solving simultaneously A = -4 , B = 6

The definite solution:

$y_t = -4 + 6(10)^t - 3t$

(e) The time path is divergent since the dominant root 10 is greater than 1. Non-oscillating since 10 is positive.

1. $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$
   $y_1 = 1$, $y_2 = 3$

   The auxiliary equation : $9r^2 + 6r + 1 = 0$ i.e. $(3r + 1)(3r + 1) = 0$
   $r = -1/3$ two equal real roots.
   The complementary function : $y_c = (A + Bt)(-1/3)^t$
   For a particular solution ,try $y_p = C + Dt$ substitute this in the original
   Equation : $9y_{t+2} + 6y_{t+1} + y_t = 2t + 1$
   $\Rightarrow 9[C + D(t+2)] + 6[C + D(t+1)] + C + Dt = 2t + 1$
   $\Rightarrow (9D + 6D + D)t + 9C + 6C + C + 18D + 6D = 2t + 1$
   $\Rightarrow$ $16Dt + 16C + 24D = 2t + 1$ equating coefficients of t and the constant terms: $16D = 2$; $D = 1/8$; $16C + 24(1/8) = 1$; $C = -2/16 = -1/8$
   Hence $y_p = (1/8)t - 1/8$
   The general solution : $y_t = y_c + y_p = (A + Bt)(-1/3)^t + (1/8)t - 1/8$
   Now use $y_1 = 1$, $y_2 = 3$ to find A and B

2. $y_{t+2} - 5y_{t+1} + 6y_t = 4^t + t$

   The auxiliary equation : $r^2 - 5r + 6 = 0$ i.e. $(r - 2)(r - 3) = 0$; $r = 2$, $r = 3$.
   The complementary function : $y_c = A(2)^t + B(3)^t$
   For a particular solution ,try $y_p = C4^t + D + Et$ substitute this in the original Equation : $y_{t+2} - 5y_{t+1} + 6y_t = 4^t + t$
   $C4^{t+2} + D + E(t+2) - 5[C4^{t+1} + D + E(t+1)] + 6(C4^t + D + Et) = 4^t + t$
   to find $C=1/2$, $D=3/4$ and $E=1/2$:
   $y_p = C4^t + D + Et = (1/2)\,4^t + ¾ + t/2$
   $y_t = y_c + y_p = A(2)^t + B(3)^t + (1/2)\,4^t + ¾ + t/2$

3. $2y_{t+2} - y_{t+1} - y_t = \sin\pi t$

   The auxiliary equation : $2r^2 - r - 1 = 0$ i.e. $(2r - 1)(r - 1) = 0$
   $r = 1/2$, $r = 1$.
   The complementary function : $y_c = A(1/2)^t + B(1)^t = A(1/2)^t + B$
   For a particular solution ,try $y_p = C\cos\pi t + D\sin\pi t$ substitute this in the original Equation : $2y_{t+2} - y_{t+1} - y_t = \sin\pi t$ to find C and D.

$2(C\cos(t+2)\pi + D\sin(t+2)\pi) - (C\cos(t+1)\pi + D\sin(t+1)\pi) - (C\cos\pi t + D\sin\pi t) = \sin\pi t$

Now $\sin(t+2)\pi = \sin\pi t$, $\cos(t+2)\pi = \cos\pi t$, $\sin(t+1)\pi = -\sin\pi t$, $\cos(t+1)\pi = -\cos\pi t$:

$2(C\cos t\pi + D\sin t\pi) - (-C\cos t\pi - D\sin t\pi) - (C\cos\pi t + D\sin\pi t) = \sin\pi t$

$2C\cos t\pi - 2D\sin t\pi = \sin\pi t$; $C = 0$; $D = -½$

$$y_t = y_c + y_p = A(1/2)^t + B - ½\sin\pi t$$