

International Institute for Technology and Management

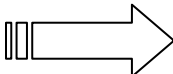


October 18th, 2005

Unit 76: Management Mathematics

Handout #5a

Difference Equations I

Topic	Interpretation
<p>Difference Equations Difference equation (or Dynamical system) is a <i>sequence of numbers</i> that generated recursively using a rule to relate each number in the sequence to previous numbers in the sequence.</p> <p><u>Examples:</u> $Y_t = a + bY_{t-1}$; a, b constants. $Y_{k+2} + Y_{k+1} - 6Y_k = 0$</p> <p>Order of Difference Equation The <i>order</i> of a dynamical system or difference equation is the difference between the largest and the smallest arguments k appearing in it.</p> <p>Find the order of the following difference equations: (1) $Y_{k-3} - 3Y_{k-4} = 0$ (2) $Y_{k+3} + aY_{k+1} = bY_{k-1} + c$ (1) $k-3 - (k-4) = -3 + 4 = 1$</p>	<p>Dynamical system is a system that changes over time. Difference equation is an equation involving differences. We can see difference equation from at least three points of views: as sequence of number, dynamical system and iterated function.</p> <p>$x = 5$ is an example of a static system because the variable x is constant over time. $x = 5t$ is an example of simple dynamic system. The system is depending on the value of the time t.</p> <p><u>Example:</u> $Y_{k+1} = aY_k + b$ has order $k+1 - k = 1$ (2) $k+3 - (k-1) = 3 + 1 = 4$</p>
<p>Solution of Difference Equations (1) First order : We have difference equation $y_k = ay_{k-1} + b$ with initial value y_0 $k = 1 : y_1 = ay_0 + b$ $k = 2 :$ $y_2 = ay_1 + b = a[ay_0 + b] + b$ $y_2 = a^2y_0 + b(1 + a)$  $k = 3 : y_3 = ay_2 + b$ $= a[a^2y_0 + b(1 + a)] + b$ $y_3 = a^3y_0 + b(1 + a + a^2)$</p>	<p>$k = n : y_n = ay_{n-1} + b$ $y_n = a[a^{n-1}y_0 + b(1 + a + a^2 + \dots + a^{n-2})] + b$ $y_n = a^n y_0 + b(1 + a + a^2 + \dots + a^{n-1})$ $1 + a + a^2 + \dots + a^{n-1}$ is a GP of first term 1, number of terms n and common ratio $r = a$: $1 + a + a^2 + \dots + a^{n-1} = a_1 \times \frac{1-r^n}{1-r} = 1 \times \frac{1-a^n}{1-a}$ However if $a = 1$ then $1 + a + a^2 + \dots + a^{n-1} = n$ $y_n = y_0 + bn$ if $a = 1$ $y_n = a^n y_0 + b(1 - a^n / 1 - a)$ if $a \neq 1$</p>

Topic	Interpretation
<p>Examples:</p> <p>(1) $y_k = 2y_{k-1} - 5 ; y_0 = y_0$</p> <p>(2) $y_k = y_{k-1} + 3 ; y_0 = 7$</p> <p>Alternative method:</p> <p>$y_k = ay_{k-1} + b$</p> <p>We find a solution which is independent of k; Let this solution be y^* then :</p> $y^* = ay^* + b \Rightarrow y^* = \frac{b}{1-a}$ <p>Now use the substitution:</p> $y_k = y^* + z_k$ $y_k = ay_{k-1} + b$ $\Rightarrow y^* + z_k = a(y^* + z_{k-1}) + b$ $\Rightarrow y^* + z_k = ay^* + az_{k-1} + b$ <p>But $y^* = ay^* + b \Rightarrow z_k = az_{k-1}$</p> $z_1 = az_0 ;$ $z_2 = az_1 = a(az_0) = a^2z_0$ $z_k = a^kz_0$ $y_k = y^* + z_k = y^* + a^kz_0$ <p>Since $y_k = y^* + z_k \Rightarrow y_0 = y^* + z_0$</p> $\Rightarrow z_0 = y_0 - y^*$ <p>Hence $y_k = y^* + a^k(y_0 - y^*)$</p>	<p>(1) $y_n = a^n y_0 + b(1 - a^n / 1 - a)$ $a = 2 ; b = - 5$</p> $y_n = 2^n y_0 - 5\left(\frac{1 - 2^n}{1 - 2}\right)$ $y_n = 2^n y_0 + 5(1 - 2^n) =$ $= 2^n y_0 + 5 - 5 \times 2^n = 5 + 2^n(y_0 - 5)$ <p>(2) $a = 1 ; b = 3 ; y_0 = 7$ $y_n = y_0 + bn = 7 + 3n$</p> <p>Alternative method solution:</p> <p>(1) $y_k = 2y_{k-1} - 5$ $a = 2 ; b = - 5$</p> $y^* = \frac{b}{1-a} = 5$ $y_n = y^* + a^n (y_0 - y^*)$ $y_n = 5 + 2^n(y_0 - 5)$ <p>(2) $y_k = y_{k-1} + 3 ; y_0 = 7$ $a = 1 ; b = 3 ; y_0 = 7$</p> <p>This says simply that each term is obtained by adding 3 to the previous one</p> $y_1 = y_0 + 3$ $y_2 = y_1 + 3 = y_0 + 2(3)$ $y_3 = y_2 + 3 = y_0 + 2(3) + 3 = y_0 + 3(3)$ $y_n = y_0 + 3n \text{ with } y_0 = 7$ $y_n = 7 + 3n$