International Institute
for Technology and Management

Unit 76: Management Mathematics

Handout #11b

Forecasting- II

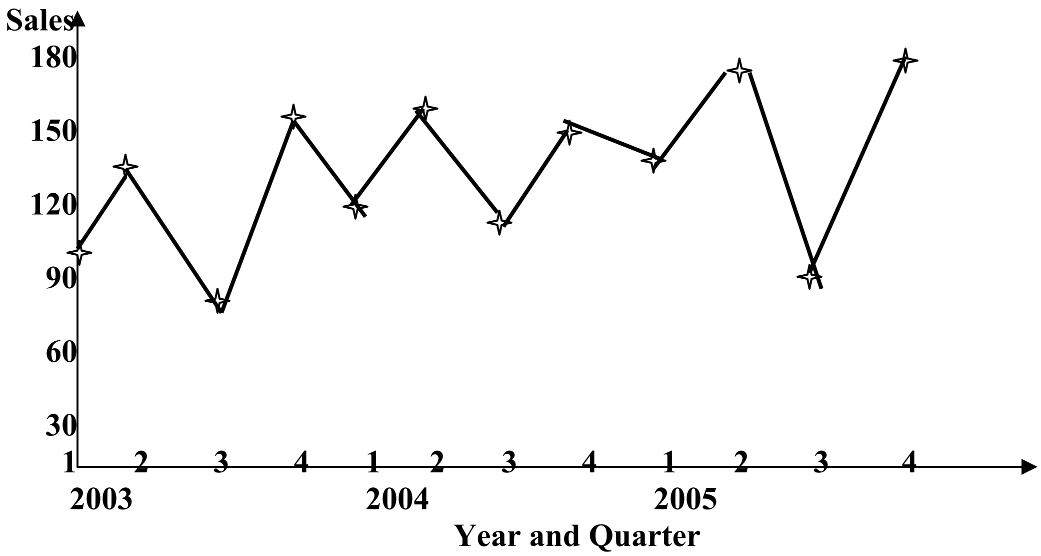
Decomposition of time series
The model : $Y_t = f(T,C,S,e)$
Y_t = actual value of time series at time t .
T = trend :long run movement represents where the series would have been in the **absence of seasonal** and random fluctuations.
C = cyclical influences (caused by long-term economic, demographic, technology)
S = seasonal influences (weather, man-made conventions: holidays, Olympics,…)
e = error (irregular influences: unexplained by T,C or S:irregular events war,earthquakes,politics,..)

Types of decomposition :

1. Additive : $Y_t = T + C + S + e$
2. Multiplicative : $Y_t = T \times C \times S \times e$

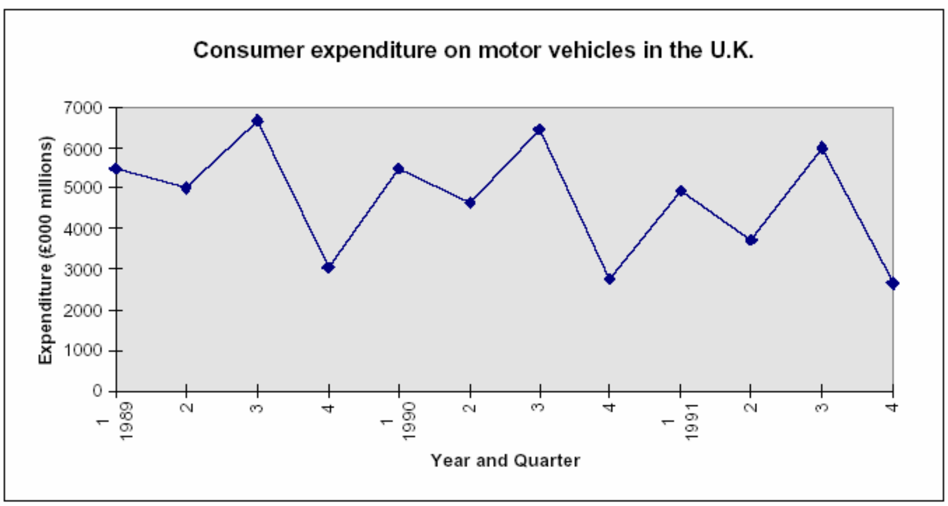
Additive or Multiplicative:
The determination whether seasonal influences are additive or multiplicative is usually evident from the graph of the data.
A multiplicative model is used if the differences of the peaks and the troughs get greater as trend increases:
Example:

Year	Quarter	Sales
2003	1	100
	2	141
	3	84
	4	**164**
2004	1	112
	2	**152**
	3	102
	4	**148**
2005	1	131
	2	160
	3	97
	4	**165**



Seasonality with peaks in quarter 4 and troughs in quarter 3 .

An additive model is used if the differences between the peaks and troughs is **constant**.
Example



Comment:

The data shows clear seasonality:

- peaks in quarter3
- troughs in quarter 4

CENTRED MOVING AVERAGE

First, we average Year 1, quarters 1-4, Then, we average Year 1, quarters 2-4 and Year 2, quarter 1. Finally, we take the *average of these two results*. This is now a kind of average taken over five successive quarters and we can place it squarely in the middle of these five quarters, against quarter 3 of year 1:

year	quarter	sales (1000s)	Four-quarter average	Centred Average
1	1	4.9		
	2	4.0		
	3	6.1		5.525
	4	6.6		
2	1	5.9	5.4	
			5.65	

The simplest estimate of trend is thus either:

- a *moving average* over an **uneven** number of periods, or
- a *centred moving average* over an even number of periods.

year	quarter	sales (1000s)	Centred Average
1	1	4.9	
	2	4.0	
	3	6.1	5.525
	4	6.6	5.8125
2	1	5.9	6.075
	2	5.3	6.2875
	3	6.9	6.425
	4	7.5	6.5
3	1	6.1	6.6375
	2	5.7	6.775
	3	7.6	
	4	7.9	

-For a 4-centered moving average, the first to Fill is the 3rd
-For a 12-centered moving average ,the first to fill is the 7th
-Formula :4-centerd: $T = 1/8\,[X_{t-2} + 2(X_{t-1} + X_t + X_{t+1}) + X_{t+2}]$

12-centerd : $T = 1/24\,[\,X_{t-6} + 2(X_{t-5} + X_{t-4} + \ldots\ldots + X_{t+5}) + X_{t+6}\,]$

This method has disadvantages:

First, the trend does not have figures for all periods, because we we lose 3 points at each end.

Secondly, each trend figure is calculated over a limited number of periods only, and all earlier data is ignored.

Finally, this method requires that we choose a suitable period as a basis for calculation, which is not always self-evident.

Decomposition Steps:

1. **Identifying Trend: This is done by a centered moving average**.

Year	Quarter	Sales	Trend: 4-Centred MA T_t
2003	1	100	
	2	141	
	3	84	123.75
	4	164	126.625
2004	1	112	130.25
	2	152	130.5
	3	102	130.875
	4	148	134.25
2005	1	131	134.625
	2	160	136.125
	3	97	
	4	165	

2. Identifying Seasonal pattern : **Multiplicative type**
   Unadjusted Seasonal fluctuation =(X_t / T_t)
3. Deseasonalized time series
   - begin by averaging all the first quarters :

Quarter	Average		Unadjusted Seasonal fluctuations		Final Seasonal fluctuations
1	(0.855 + 0.973)/2	=	0.914	x 4/4.0105	0.911
2	(1.164+ 1.175)/2	=	1.1695		1.166
3	(0.678+0.779)/2	=	0.7285		0.726
4	(1.295 + 1.102)/2	=	1.1985		1.195
			----------		--------
	Sum =		4.0105		4.0

-Adjust the seasonal so that they equal to 4 (the number of periods per season):
multiply each of the quarters average by **4 / total of averages**

Year	Quarter	Sales X_t	4-Centred MA T_t	Raw Seasonal fluctuation X_t/T_t	Final Seasonal fluctuation S_t	Deseasonlized Sales X_t / S_t
2003	1	100			0.911	
	2	141			1.166	
	3	84	123.75	0.678	0.726	115.7
	4	164	126.625	1.295	1.195	137.23
2004	1	112	130.25	**0.855**	0.911	…….
	2	152	130.5	1.164	1.166	………..
	3	102	130.875	0.779	0.726	……….
	4	148	134.25	1.102	1.195	………
2005	1	131	134.625	**0.973**	0.911	………
	2	160	136.125	1.175	1.166	…….
	3	97	133.968		0.726	…….
	4	165	134.742		1.195	……..

-Forecasting :

Develop a forecast for four quarters (starting with quarter 1 in 2006), using a four – quarter moving average based upon the centered moving average sales series. First, you will have to "fill in" the data for quarters 3 and 4 in 2005. Just make each of these quarters the average of the previous four quarters.

Continue this method to make your forecast for quarters 1 through 4 in 2006. Note that you will be using some of your "early" forecasts as the basis for your "later" forecasts. Once you compute the four-quarter moving average forecast, you will then need to "**re-seasonalize**" your forecast. This is done by **multiplying** the forecast for each quarter by the appropriate average seasonal adjustment factor.

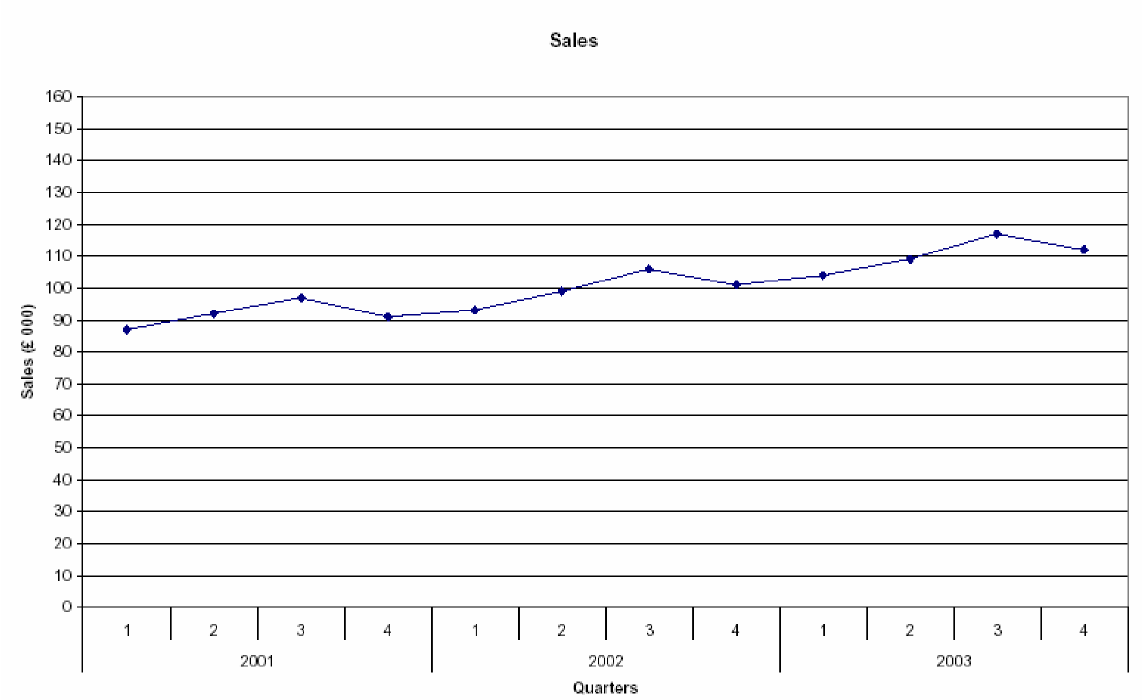
		X_t	Forecasted Deseasonlized	Final seasonal	re-seasonalized
2006	1	131	134.865	0.911	122.047
	2	160	134.929	1.166	157.327
	3	97	134.626	0.726	97.738
	4	165	134.790	1.195	161.074

Example

Day	Volume	Moving Total	Centered MA_7	Seasonal Index = $Volume/MA_7$
Tues	67			
Wed	75			
Thur	82			
Fri	98		71.86	98/71.86=1.36
Sat	90		70.86	90/70.86=1.27
Sun	36		70.57	36/70.57=0.51
Mon	55	503	71.00	55/71.00=0.77
Tues	60	496	71.14	60/71.14=0.84
Wed	73	494	70.57	73/70.57=1.03
Thur	85	497	71.14	85/71.14=1.19
Fri	99	498	70.71	99/70.71=1.40
Sat	86	494	71.29	86/71.29=1.21
Sun	40	498	71.71	40/71.71=0.56
Mon	52	495	72.00	52/72.00=0.72
Tues	64	499	71.57	64/71.57=0.89
Wed	76	502	71.86	76/71.86=1.06
Thur	87	504	72.43	87/72.43=1.20
Fri	96	501	72.14	96/72.14=1.33
Sat	88	503		
Sun	44	507		
Mon	50	505		

Example: Additive model

Year	Quarter	Exports	Trend CMA_4	Seasonal Deviations
2001	1	87		
	2	92		
	3	97	92.5	4.5
	4	91	94.125	-3.125
2002	1	93	96.125	-3.125
	2	99	98.5	0.5
	3	106	101.125	4.875
	4	101	103.75	-2.75
2003	1	104	106.375	-2.375
	2	109	109.125	-0.125
	3	117		
	4	112		



Upward trend, seasonality with peaks in quarter 3 and troughs in quarter 1.
The seasonal deviation column suggests that an additive model is being used. The chart shows fairly constant deviations from the underlying trend independent of time.

-Again (as in the multiplicative model) average all the first quarters :
for Q1 : [3.125 +(-2.375)]/2 = -2.75

- Adjust the seasonal so that they equal to 4 (the number of periods per season):
  Subtract their average from each of the quarters average :
  Adjustment = (average Q1 + Average Q2 + Average Q3 + averageQ4)/4
  = (-2.75 +0.188+4.688 -2.938)/4 = -0.203
- For Q1 : Adjusted seasonal = average Q1 –(- 0.203)= -2.75 + 0.203 = -2.547

	2001	2002	2003	Mean SD	Adjusted SD
Q1		-3.125	-2.375	-2.75	-2.547
Q2		0.5	-0.125	0.188	0.391
Q3	4.5	4.875		4.688	4.891
Q4	-3.125	-2.75		-2.938	-2.734
Total				-0.813	0
Adjustment				-0.203	

Forecasting :

Develop a forecast for four quarters (starting with quarter 1 in 2004), using a four – quarter moving average based upon the centered moving average sales series. First, you will have to "fill in" the data for quarters 3 and 4 in 2003. Just make each of these quarters the average of the previous four quarters.

Continue this method to make your forecast for quarters 1 through 4 in 2004. Note that you will be using some of your "early" forecasts as the basis for your "later" forecasts. Once you compute the four-quarter moving average forecast, you will then need to "**re-seasonalize**" your forecast. This is done by **Adding** the forecast for each quarter by the appropriate average seasonal adjustment factor.

Forecast = Trend + Adjusted Seasonal

Example 2 :Television Sales

year	quarter	sales (1000s)	Centred Average	Seasonal variation
1	1	4.9		
	2	4.0		
	3	6.1	5.525	6.1-5.525=0.575
	4	6.6	5.8125	6.6-5.8125=0.7875
2	1	5.9	6.075	5.9-6.075 =-0.175
	2	5.3	6.2875	-0.9875
	3	6.9	6.425	0.475
	4	7.5	6.5	1
3	1	6.1	6.6375	-0.5375
	2	5.7	6.775	-1.075
	3	7.6		
	4	7.9		

It is only a first estimate, however, because it is 'contaminated' by the residual random variation R. We eliminate this by taking an average across all cases of each season in our data, producing *seasonal indexes*.

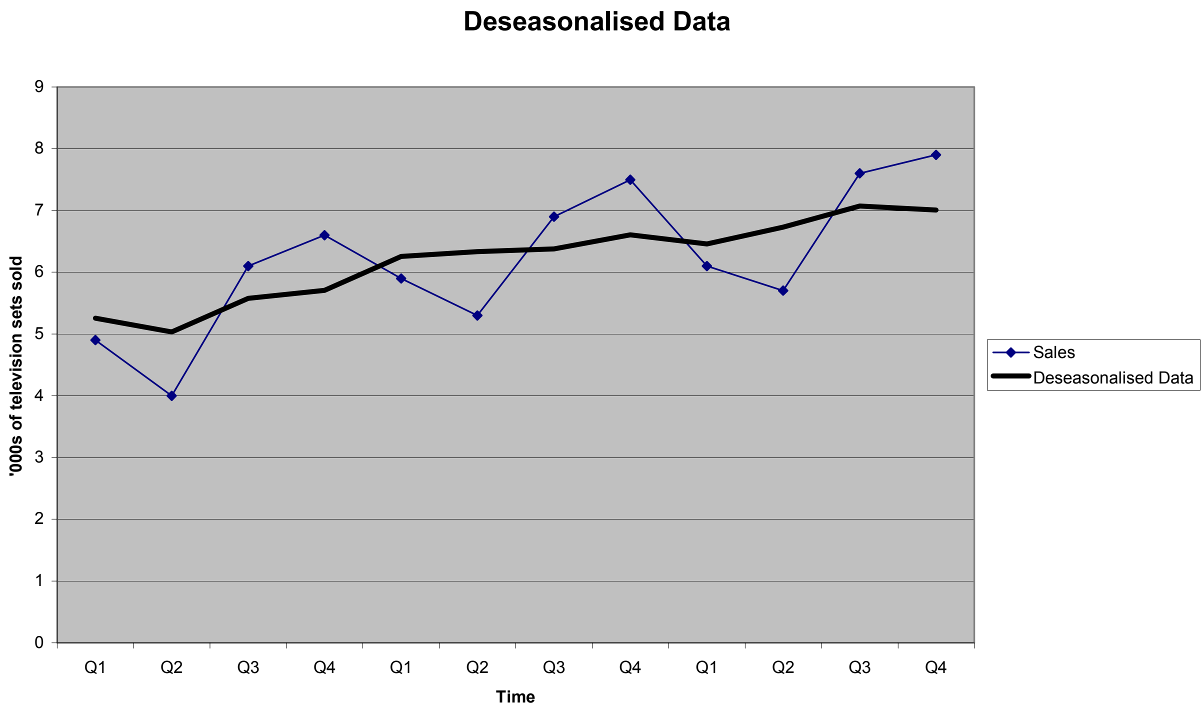
That is, to calculate the seasonal index for quarter 1, we take the average of the seasonal variation in quarter 1, year 1, the seasonal variation in quarter 1, year 2, and so on.

The seasonal index is calculated, for each quarter, by averaging all the seasonal variations for that quarter.This provides an estimate of the true S_t for each quarter.

Quarter	Seasonal index
1	(–0.175–0.5375)/2= –0.35625
2	(–0.9875–1.075)/2= –1.03125
3	(0.575+0.475)/2 = 0.525
4	(0.7875+1)/2 = 0.89375

Finally, to deseasonalise the series, subtract the seasonal index from each of the corresponding sales figures to produce an estimate of $T_t + R_t$, the trend plus the residual or random component, without the seasonal component. This is called the *deseasonalised* data.

year	quarter	sales (1000s)	Seasonal Index	Deseasonalised data
1	1	4.9	-0.35625	4.9-(-0.35635)=5.25635
	2	4.0	-1.03125	4.0-(-1.03125)= 5.03125
	3	6.1	0.525	6.1-0.525 = 5.575
	4	6.6	0.89375	6.6-0.89375 =5.70625
2	1	5.9	-0.35625	6.25625
	2	5.3	-1.03125	6.33125
	3	6.9	0.525	6.375
	4	7.5	0.89375	6.60625
3	1	6.1	-0.35625	6.45625
	2	5.7	-1.03125	6.73125
	3	7.6	0.525	7.075
	4	7.9	0.89375	7.00625



Estimating a linear trend

In the last section, we deseasonalised the data assuming:

(1) a moving average trend

(2) an additive model

As the notes indicated, there are other ways to estimate a trend and other models. We will only study one of them: a *linear* trend.

A linear trend means that we think there is an underlying relation between the observations and time which is given by the linear equation

$$T_t = a + bt$$

where a and b are constants. As before, the actual observations X_t are then given by

$$X_t = T_t + S_t + R_t$$

Ignoring cyclical variation and assuming and additive model.
You should remember from the mathematics you have learned that

a is the intercept; the value of T_t at time $t = 0$

b is the slope or steepness of the graph

You should also know how to **estimate a trend line by hand**, **graphically**. Here are the relevant steps:

(1) draw a graph of sales against time. This does not have to be a line graph; it can be a *scatter plot* as shown below.

(2) draw a line, by hand, which best seems to fit the points; it should run between the points as shown below also.

(3) The *intercept a* is can simply be read off from the *T* axis. On the diagram below you can see that this is about 4.7

(4) To calculate *b* you need to measure the value of *X*, as estimated by your line, at some other point than the axis. Preferably this should be as far away from the *T*-axis as possible. Thus we might take the point $t = 12$, $X_t = 7.5$. We can now use the formula for the line itself to estimate *b*; according to this formula, T_t is given by

$$a + bt = 4.7 + bt$$

Now substitute t=12, X=7.5 into this equation to give

$$7.5 = 4.7 + b\times 12$$

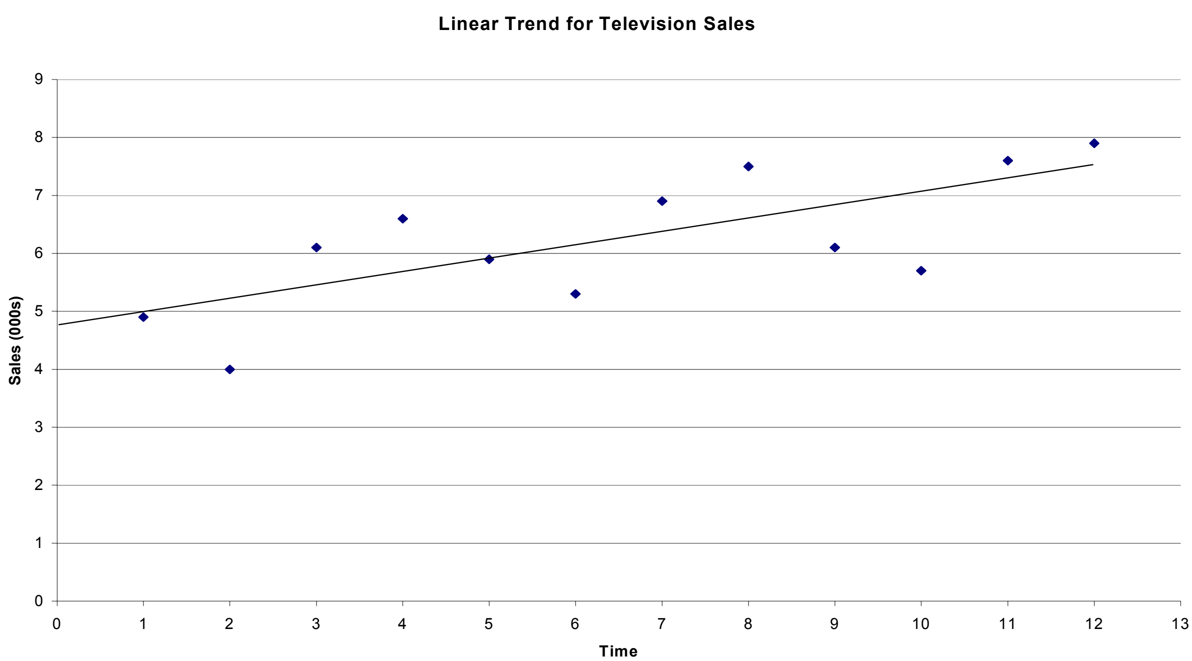
from which

$$12\times b = 7.5 - 4.7 = 2.8$$

$$\text{and so } b = 2.8/12 = 0.2333$$

Thus the estimate equation of the line is

$$T_t = 4.7 + 0.2333t$$



Now you can use the trend line to Forecast for quarters in Year 4 :
quarter 1 in Year 4 : when t = 13

$T_t = 4.7 + 0.2333t = 4.7 + 0.2333(13) = 7.73$
Foe quarters 2,3,qnd 4 use t =14,15 and 16 respectively.
Exact Equation of the Trend :
Another way to find the equation of the trend is by using the Least squares method :

$$Y = a + bt \;;\; b = \frac{n\sum tY - \sum t \sum Y}{n\sum t^2 - \left(\sum t\right)^2} \quad \text{and} \quad a = \frac{\sum Y - b\sum t}{n}$$

Example

Week t	sales Y	tY	t^2
1	700	700	1
2	724	1448	4
3	720	2160	9
4	728	2912	16
5	740	3700	25
6	742	4452	36
7	758	5306	49
8	750	6000	64
9	770	6930	81
10	775	7750	100
$\sum$ 55	7407	41358	385

$$b = \frac{n\sum tY - \sum t \sum Y}{n\sum t^2 - \left(\sum t\right)^2} = \frac{10(41358) - (55)(7407)}{10(385) - (55)^2} = 7.5$$

$$a = \frac{\sum Y - b\sum t}{n} = \frac{7407 - (7.5)(55)}{10} = 699.45$$

Trend : Y = 699.45 + 7.5t

Forecast for week 11; t = 11
Y = 699.45 + 7.5(11) = 782

Forecast for week 12; t = 12
Y = 699.45 + 7.5(12) = 789.5