International Institute
for Technology and Management

Unit 76: Management Mathematics **Handout #10**

Constrained Optimization

Critical points

Let f(x,y,z…) be a function of several variables. The Critical points are obtained by Solving $f_1 = \frac{\partial f}{\partial x} = 0; f_2 = \frac{\partial f}{\partial y} = 0; f_3 = \frac{\partial f}{\partial z} = 0, \ldots$..

Example : Find the critical point(s) of $f(x,y) = y^3 + 3xy - x^3$

$f_1 = 3y - 3x^2 = 0 \Rightarrow y = x^2$; $f_2 = 3y^2 + 3x = 0 \Rightarrow y^2 + x = 0$

$\Rightarrow (x^2)^2 + x = 0 \Rightarrow x^4 + x = 0 \Rightarrow x(x^3 + 1) = 0$

$\Rightarrow x(x+1)(x^2 - x + 1) = 0 \Rightarrow x = 0$ or $x = -1$ $x = 0 \Rightarrow y = x^2 = 0 \Rightarrow (0, 0)$

$x = -1 \Rightarrow y = x^2 = 1 \Rightarrow (-1,1)$

Nature of Critical points :

1. Maximum /Minimum:

$(f_{11})(f_{22}) - f_{12}^2 > 0$

If $f_{11} > 0 \Rightarrow$ Minimum

If $f_{11} < 0 \Rightarrow$ Maximum

2. Saddle point :

$(f_{11})(f_{22}) - f_{12}^2 < 0$

Example: Find and classify the critical points of the following function:

$f(x,y) = x^2 + y^2$

$f_1 = 2x = 0 \Rightarrow x = 0$; $f_2 = 2y = 0 \Rightarrow y = 0$ $(0, 0) \Rightarrow$ is the critical point .

$f_{11} = 2$, $f_{22} = 2$; $f_{12} = 0$

$\Rightarrow (f_{11})(f_{22}) - f_{12}^2 = 4 > 0$ there is a maximum/minimum

Since $f_{11} = 2 > 0 \Rightarrow (1,1)$ minimizes f .

Constrained Optimization:

Suppose f(x,y,z,…) has to be minimized or maximized subject to the constraint g(x,y,z,…) = 0 .

1. Using substitution:

Example : Minimize $f = x^2 + 4y^2$ subject to the constraint $x - y = 10$

Express one of the variables in terms of the others : $y = x - 10$

Substitute in the objective function : $f = x^2 + 4(x - 10)^2 = 5x^2 - 80x + 400$

$f'(x) = 0 \Rightarrow 10x - 80 = 0 \Rightarrow x = 8$ & $y = x - 10 = -2$; $(x,y) = (8, -2)$.

When substitution is difficult to solve ,we use the method of

2. The Lagrange Multipliers:

1.)Set: $L(x,y,z,\ldots..\lambda) = f(x,y,z,\ldots) - \lambda g(x,y,z,\ldots.)$
2.)Find x and y as solutions of:

$$\frac{\partial L}{\partial x} = 0\ ,\ \frac{\partial L}{\partial y} = 0\ ,\ \frac{\partial L}{\partial z} = 0, \ldots,\ \frac{\partial L}{\partial \lambda} = g(x,y,z,\ldots) = 0$$

Example:

Use the Lagrange multiplier to find the values of x and y which maximizes $160x - 3x^2 - 2xy - 2y^2 + 120y - 18$ subject to the constraint $x + y = 34$

$f(x,y) = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18$; $g(x,y) = x + y - 34 = 0$

$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$L = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18 - \lambda(x + y - 34)$

$$\frac{\partial L}{\partial x} = 160 - \lambda - 6x - 2y = 0 \text{ -------(1)}$$

$$\frac{\partial L}{\partial y} = -2x - 4y + 120 - \lambda = 0 \text{ ----(2)}$$

$$\frac{\partial L}{\partial \lambda} g(x,y) = x + y - 34 = 0 \text{ ---------(3)}$$

Eliminate λ from (1) & (2); (1): $\lambda = 160 - 6x - 2y$;(2): $\lambda = -2x - 4y + 120$

$\lambda = \lambda \Rightarrow 160 - 6x - 2y = -2x - 4y + 120 \Rightarrow 4x + 2y + 40 = 0$

$\Rightarrow 2x + y + 20 = 0$ -----(4) ; (3) & (4) :

$x + y - 34 = 0$; $2x + y + 20 = 0$ solving simultaneously:

$x = 18$, $y = 16$; substitute these in $\lambda = 160 - 6x - 2y \Rightarrow \lambda = 20$.

The constrained maximum $f(18,16) = 2722$

NB : If no constraint is imposed then the maximum is at (20,20)

$f_1 = 160 - 6x - 2y = 0$

$f_2 = -2x - 4y + 120 = 0$ Solving simultaneously:

$x = 20$, $y = 20$; $f_{11} = -6, f_{22} = -4; f_{12} = -2 \Rightarrow (f_{11})(f_{22}) - f_{12}^2 = 20 > 0$

Since $f_{11} = -6 < 0 \Rightarrow$ (20,20) maximizes f .

Multiple constraints

Suppose f(x,y,z,…) has to be minimized or maximized subject to the constraint $g_1(x,y,z,\ldots) = 0$, $g_2(x,y,z,\ldots) = 0$, $g_3(x,y,z,\ldots) = 0$,…… Then use :

$$L(x,y,z,\ldots..\lambda) = f(x,y,z,\ldots) - \lambda_1 g_1(x,y,z,\ldots.) - \lambda_2 g_2(x,y,z,\ldots.) - \ldots\ldots\ldots$$

$$\frac{\partial L}{\partial x} = 0\ ,\ \frac{\partial L}{\partial y} = 0\ ,\ \frac{\partial L}{\partial z} = 0, \ldots,\ \frac{\partial L}{\partial \lambda_1} = g_1(x,y,z,\ldots) = 0\ ,\ \frac{\partial L}{\partial \lambda_2} = g_2(x,y,z,\ldots) = 0\ ,\ldots..$$

Meaning of the multiplier

The Lagrange multiplier is an *artificial* variable added for *computational convenience*,suppose the optimum value is at the point **P(x,y)** , the constraint function **g(P)** can be thought of as "competing" with the desired function **f(P)** to "pull" the point **P** to its minimum or maximum. The Lagrange multiplier **λ** can be thought of as a measure of how hard **g(P)** has to pull in order to make those "forces" balance out on the constraint surface.

Economic Interpretation

The economic interpretation of the multiplier as a 'shadow price': For example, in the application for a firm maximizing profits, it tells us how valuable another unit of input would be to the firm's profits, or how much the maximum value changes for the firm when the constraint is relaxed. In other words, it is the maximum amount the firm would be willing to pay to acquire another unit of input.

The values λ have an important economic interpretation: If the right hand side **b** of Constraint *i* is increased by Δ, then the optimum objective value increases by approximately $\lambda\ \Delta$. In particular,

Consider the problem **Maximize *p*(*x*) subject to *g*(*x*)=*b***,
where *p*(*x*) is a profit to maximize and *b* is a limited amount of resource. Then, the optimum Lagrange multiplier **λ** is the marginal value of the resource. Equivalently, if ***b*** were increased by Δ, profit would increase by $\lambda\,\Delta$. This is an important result to remember. Similarly, if Minimize *c*(*x*) subject to *d*(*x*)=*b*, represents the minimum cost *c*(*x*) of meeting some demand *b*, the optimum Lagrange multiplier λ is the marginal cost of meeting the demand. if ***b*** were increased by Δ, cost would increase by $\lambda\,\Delta$

Example: Minimize **f =** $400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$
Subject to : $x + y + 1.5z \geq 1.2$ & $x + y + z \geq 1$
What would you expect the minimum to be if the first constraint is $x + y + 1.5z \geq 1.25$;

$$L = 400x^2+800y^2+200xy+1600z^2+400yz - \lambda_1(x+y+1.5z-1.2) - \lambda_2(x + y + z - 1)$$

$$\frac{\partial L}{\partial x} = 0 \;, 800x + 200y - \lambda_1 - \lambda_2 = 0 \;; \quad \frac{\partial L}{\partial y} = 0 \;; 1600y+200x+400z-\lambda_1-\lambda_2 = 0$$

$$\frac{\partial L}{\partial z} = 0;\; 3200z+400y-1.5\lambda_1-\lambda_2 = 0; \quad \frac{\partial L}{\partial \lambda_1} = x+y+1.5z - 1.2 = 0 \;,$$

$\frac{\partial L}{\partial \lambda_2}$ = x + y + z- 1 = 0 ; Solving : **x = 0.5 , y = 0.1 , z = 0.4 ,** λ_1 **= 1800 ,** λ_2 **= - 1380** The corresponding objective function :

$$f = 400(0.5)^2 + 800(0.1)^2 + 200(0.5)(0.1)+1600(0.4)^2 +400(0.1)(0.4)= \mathbf{390}$$

first constraint is x + y + 1.5z = 1.25 is increased by 1.25 – 1.2 **= 0.05**

Since λ_1 is the multiplier associated with the first constraint then the objective function is increases **0.05**λ_1 **= (0.05)(1800) = 90**

The expected value becomes **390 + 90 = 480**

Binding Constraints

A **binding constraint** (*active*) is a resource which is completely exhausted. e.g. use all resources available.

In this case the multiplier $\lambda \succ$ **0** (*strictly positive*).

A **non-binding constraint** (*not active*) is a resource that does not limit the improvement in the objective function, e.g. have "left over" resources.

In this case the multiplier λ **= 0**.

Example: L = X + 4Y - λ_1**(X + Y – 10) -** λ_2**(2X + Y -12)**