



Tutoring Sheet #5 - Solution

Unit 76: Management Mathematics –Difference Equations

1. Solve the following Difference Equations:

a. $y_t = 3 y_{t-1} + 4$; $y_0 = 2$; $y_t = 3^t y_0 + 4\left(\frac{1-3^t}{1-3}\right) = 2(3^t) - 2(1-3^t)$

b. $y_{t+3} = 2 y_{t+2} - 5$; $y_t = 2^{t-2} y_2 - 5\left(\frac{1-2^{t-2}}{1-2}\right) = 2^{t-2} y_2 + 5(1-2^{t-2})$

c. $y_{t+1} = y_t + 13$; $y_0 = 5$; $y_t = y_0 + 13t = 5 + 13t$

2. Solve the following difference equations and describe the solution series:

a. $y_{t+2} - y_{t+1} - 2y_t = 3$ with $y_0 = 2$; $y_1 = 2$

The auxiliary equation : $r^2 - r - 2 = 0 \Rightarrow (r + 1)(r - 2) = 0$
 $r = -1$; $r = 2$ two distinct real roots.

The complementary function : $y_c = A(-1)^t + B(2)^t$

The equation has the form : $y_{t+2} + ay_{t+1} + by_t = 0$

Where $a + b = -1 - 2 = -3 \neq -1$

For a particular solution , $y_p = \frac{c}{1+a+b} = -3/2$

The general solution : $y_t = y_c + y_p = A(-1)^t + B(2)^t - 3/2$

Now use $y_0=2$, $y_1=2$ to find A and B

$y_0=2 \Rightarrow A(-1)^0 + B(2)^0 - 3/2 = 2 \Rightarrow A + B = 7/2$

$y_1=2 \Rightarrow A(-1)^1 + B(2)^1 - 3/2 = 2 \Rightarrow -A + 2B = 7/2$

Solving for A and B simultaneously: $A = 7/6$; $B = 7/3$

The general solution : $y_t = (7/6)(-1)^t + (7/3)(2)^t - 3/2$

b. $4y_{t+2} + 4y_{t+1} + y_t = 2+3t$ with $y_1 = 1 ; y_2 = 2$

The auxiliary equation : $4r^2 + 4r + 1 = 0 \Rightarrow (2r + 1) (2r + 1) = 0$
 $r = -1/2$ two equal real roots.

The complementary function : $y_c = (A + Bt)(-1/2)^t$

For a particular solution , $y_p = C + Dt$ substitute this in the original

Equation : $4y_{t+2} + 4y_{t+1} + y_t = 2 + 3t$

$$\Rightarrow 4[C + D(t+2)] + 4[C + D(t+1)] + C + Dt = 2 + 3t$$

$$\Rightarrow (4D + 4D + D)t + 4C + 4C + C + 8D + 4D = 2 + 3t$$

$\Rightarrow 9Dt + 9C + 12D = 2 + 3t$ equating coefficients of t and the constant terms:
 $9D = 3 ; D = 1/3 ; 9C + 12(1/3) = 2 ; C = -3$

Hence $y_p = (1/3)t - 3$

The general solution : $y_t = y_c + y_p = (A + Bt)(-1/2)^t + (1/3)t - 3$

Now use $y_1=1 , y_2= 2$ to find A and B :

$$y_1=1 \Rightarrow (A + B(1))(-1/2)^1 + (1/3)(1) - 3 = 1 \Rightarrow (A+B)(-1/2) = 11/3$$

$$y_2= 2 \Rightarrow (A + B(2))(-1/2)^2 + (1/3)(2) - 3 = 2 \Rightarrow A/4 + B/2 = 13/3$$

Solving for A and B simultaneously: $A = -32 ; B = 74/3$

The general solution : $y_t = (-32 + (74/3)t)(-1/2)^t + (1/3)t - 3$

c. $y_t + 4y_{t-2} = 15$ with $y_0 = 12 ; y_1 = 11$

The auxiliary roots : $r^2 + 4 = 0 \Rightarrow r = -2i , r = +2i$

The complementary function: $y_c = (\sqrt{4})^t (A \cos \alpha t + B \sin \alpha t)$

$$\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}(0) = \frac{\pi}{2} ; y_t = 2^t (A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t)$$

The equation has the form : $y_t + ay_{t-1} + by_{t-2} = c$

Where $a + b = 0 + 4 = 4 \neq -1$

For a particular solution , $y_p = \frac{c}{1 + a + b} = 3$

The general solution: $y_t = y_c + y_p = 2^t (A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t) + 3$

$$y_0 = 12 \Rightarrow 2^0 (A \cos 0 + B \sin 0) + 3 = 12 \Rightarrow A + 3 = 12 \Rightarrow A = 9$$

$$y_1 = 11 \Rightarrow 2^1 (A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2}) + 3 = 11 \Rightarrow 2B + 3 = 11 \Rightarrow B = 4$$

The general solution :

$$y_t = 2^t (9 \cos \frac{\pi}{2} t + 4 \sin \frac{\pi}{2} t) + 3$$

d. $y_{t+2} + 4 y_{t+1} + 5 y_t = 20$

The auxiliary equation : $r^2 + 4r + 5 = 0 \Rightarrow 2$ complex roots $-2 \pm 4i$

The complementary function: $y_c = (\sqrt{5})^t (A \cos \alpha t + B \sin \alpha t)$

$$\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}\left(\frac{-4}{2\sqrt{5}}\right) = 153.43^\circ \times \frac{\pi}{180} = 0.85\pi$$

$$; y_t = (\sqrt{5})^t (A \cos 0.85\pi t + B \sin 0.85\pi t)$$

The equation has the form : $y_{t+2} + ay_{t+1} + by_t = 0$

Where $a + b = 4 + 5 = 9 \neq -1$

$$\text{For a particular solution , } y_p = \frac{c}{1+a+b} = 2$$

The general solution: $\mathbf{y_t = y_c + y_p}$

$$\mathbf{y_t = (\sqrt{5})^t (A \cos 0.85\pi t + B \sin 0.85\pi t) + 2}$$

e. $y_t + 4 y_{t-2} = 13 - 5t$ with $y_0 = 6 ; y_1 = 8$

The auxiliary roots : $r^2 + 4 = 0 \Rightarrow r = -2i , r = +2i$

The complementary function: $y_c = (\sqrt{4})^t (A \cos \alpha t + B \sin \alpha t)$

$$\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}(0) = \frac{\pi}{2} ; y_t = 2^t (A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t)$$

For a particular solution , $y_p = C + Dt$ substitute this in the original

Equation : $y_t + 4y_{t-2} = 13 - 5t$

$$C + Dt + 4[C + D(t-2)] = 13 - 5t \Rightarrow 5Dt + 5C - 8D = 13 - 5t$$

$$\Rightarrow D = -1 ; C = 1 \Rightarrow y_p = 1 - t$$

The general solution: $\mathbf{y_t = y_c + y_p = 2^t (A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t) + 1 - t}$

$$y_0 = 6 \Rightarrow 2^0 (A \cos 0 + B \sin 0) + 1 - 0 = 12 \Rightarrow A + 1 = 6 \Rightarrow A = 5$$

$$y_1 = 8 \Rightarrow 2^1 (A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2}) + 1 - 1 = 11 \Rightarrow 2B = 11 \Rightarrow B = 11/2$$

The general solution :

$$\mathbf{y_t = 2^t (5 \cos \frac{\pi}{2} t - \frac{11}{2} \sin \frac{\pi}{2} t) + 1 - t}$$

3. Consider the following difference equation:

$$y_t + 4y_{t-2} = 22 + 5t \text{ with } y_0 = 1 ; y_1 = 3$$

Solve the above difference equation for y , graph the Solution and describe the graph in words.

The auxiliary roots : $r^2 + 4 = 0 \Rightarrow r = -2i, r = +2i$

The complementary function: $y_c = (\sqrt{4})^t (A \cos \alpha t + B \sin \alpha t)$

$$\alpha = \cos^{-1}\left(\frac{-a}{2\sqrt{b}}\right) = \cos^{-1}(0) = \frac{\pi}{2} ; y_t = 2^t \left(A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t \right)$$

For a particular solution, $y_p = C + Dt$ substitute this in the original

Equation : $y_t + 4y_{t-2} = 22 + 5t$

$$C + Dt + 4[C + D(t-2)] = 22 + 5t \Rightarrow 5Dt + 5C - 8D = 22 + 5t$$

$$\Rightarrow D = 1 ; C = 6 \Rightarrow y_p = 6 + t$$

The general solution: $y_t = y_c + y_p = 2^t \left(A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t \right) + 6 + t$

$$y_0 = 1 \Rightarrow 2^0 (A \cos 0 + B \sin 0) + 6 + 0 = 1 \Rightarrow A + 6 = 1 \Rightarrow A = -5$$

$$y_1 = 3 \Rightarrow 2^1 \left(A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2} \right) + 6 + 1 = 3 \Rightarrow 2B + 7 = 3 \Rightarrow B = -3$$

The general solution :

$$y_t = 2^t \left(-5 \cos \frac{\pi}{2} t - 3 \sin \frac{\pi}{2} t \right) + 6 + t$$