International Institute for
Technology and Management

**November 18, 2008** **Duration: 1.5 hours**

# VERSION I

## Unit 76: Management Mathematics Assignment – 2

**Answer all the following questions:**

**1.** Solve the following difference equation: **(5 Marks)**

$$y_{t+2} - 2y_{t+1} + 2y_t = t - 1 \quad \text{with} \quad y_0 = 0 \text{ and } y_1 = 3$$

The auxiliary roots : $r^2 - 2r + 2 = 0 \Rightarrow$ r = $1 - i$ , r $= 1 + i$

The complementary function: $y_c = (\sqrt{2})^t (A\cos\alpha\, t + B\sin\alpha\, t)$

$\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$ ; $y_t = (\sqrt{2})^t (A\cos\frac{\pi}{4} t + B\sin\frac{\pi}{4} t)$

For a particular solution, $y_p$ = C + Dt substitute this in the original
Equation: $y_{t+2} - 2y_{t+1} + 2y_t = t - 1$

C+D(t+2) - 2[C+D(t+1)]+2(C+Dt) =t – 1

$\Rightarrow$ Dt + C = t – 1 $\Rightarrow$ D = 1 ; C = -1 $\Rightarrow$ **$y_p$ = t – 1**

The general solution:

$$\mathbf{y_t = y_c + y_p} = (\sqrt{2})^t (A\cos\frac{\pi}{4} t + B\sin\frac{\pi}{4} t) + \mathbf{t - 1}$$

$y_0 = 0 \Rightarrow (\sqrt{2})^0 (A\cos 0 + B\sin 0) + 0 - 1 = 0 \Rightarrow A - 1 = 0 \Rightarrow A = 1$

$y_1 = 3 \Rightarrow (\sqrt{2})^1 (A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}) + 1 - 1 = 3$

$\Rightarrow \sqrt{2}\,(A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2}) = 3 \Rightarrow A + B = 3 \Rightarrow B = 2$

The general solution :

$$\mathbf{y_t} = (\sqrt{2})^t (\cos\frac{\pi}{4} t + 2\sin\frac{\pi}{4} t) + \mathbf{t - 1}$$

**2.** For some **0 < c < 1 , w>0**, the following questions relate to income $Y_t$ ,consumption $C_t$ ,production $Q_t$ ,Investment $I_t$ and government spending $G_t$ at time t:

$$\mathbf{Y_t = C_t + I_t + G_t}$$
$$\mathbf{C_t = C_0 + cY_{t-1}}$$
$$\mathbf{I_t = I_0 + w(C_t - C_{t-1})}$$
$$\mathbf{G_t = G_0}$$

(a) Derive a second order difference equation for Y in terms of **$c$, $w$, $I_0$, $C_0$ and $G_0$** . **(3 Marks)**

The required equation should be :

$$Y_t - c(1+w)Y_{t-1} + cwY_{t-2} = C_0 + I_0 + G_0$$

$$Y_t = C_t + I_t + G_t$$

$$Y_t = C_0 + cY_{t-1} + I_0 + w(C_t - C_{t-1}) + G_0$$

$$Y_t - cY_{t-1} - w(C_t - C_{t-1}) = C_0 + I_0 + G_0 \quad \text{-------(1)}$$

We need to get rid of **$(C_t - C_{t-1})$:**

$$C_t = C_0 + cY_{t-1}$$

$$C_{t-1} = C_0 + cY_{t-2}$$

$$C_t - C_{t-1} = C_0 + cY_{t-1} - C_0 - cY_{t-2} = cY_{t-1} - cY_{t-2}$$

**(1)** becomes :

$$Y_t - cY_{t-1} - w(cY_{t-1} - cY_{t-2}) = C_0 + I_0 + G_0$$

$$Y_t - c(1+w)Y_{t-1} + cwY_{t-2} = C_0 + I_0 + G_0$$

(b) Show that for the time path to oscillate,the following relation holds : **$cw^2 + 2(c-2)w + c < 0$** **(3 Marks)**

The auxiliary equation : $r^2 - c(1+w)r + cw = 0$

To be oscillating it has to have complex roots

i.e. **$c^2(1+w)^2 - 4cw < 0$ dividing by c (being > 0):**

$$c(1+w)^2 - 4w < 0 \Rightarrow c + 2cw + cw^2 - 4w < 0$$

$$cw^2 + 2(c-2)w + c < 0$$

(c) Suppose now **$c = \frac{1}{8}$ , $w = 1$** and that **$I_0 + C_0 + G_0 = 40$**

Solve the difference equation obtained in (a) for **$Y_0 = 65$ , $Y_1 = 64.5$** **(5 Marks)**

$r^2 - (1/4)r + 1/8 = 0 \Rightarrow 8r^2 - 2r + 1 = 0$

$b^2 - 4ac = 4 - 4(8)(1) = -28 < 0$ complex roots , $\sqrt{b} = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}}$

The complementary function: $y_c = (\frac{1}{2\sqrt{2}})^t (A\cos\alpha\, t + B\sin\alpha\, t)$

$$y_c = (\frac{1}{2\times 2^{1/2}})^t (A\cos\alpha\, t + B\sin\alpha\, t) = 2^{\frac{-3}{2}t} \;(A\cos\alpha\, t + B\sin\alpha\, t)$$

$$\alpha = \cos^{-1}(\frac{-a}{2\sqrt{b}}) = \cos^{-1}(\frac{1/4}{2(1/2\sqrt{2})}) = \cos^{-1}(\frac{\sqrt{2}}{4}) = 69.29^\circ = 0.38\pi \;;$$

$$Y_c = 2^{-\frac{3}{2}t} (A\cos 0.38\pi t + B\sin 0.38\pi t)$$

$y_p = \dfrac{c}{1+a+b} = \dfrac{40}{7/8} = 45.71$

$y_t = 2^{-\frac{3}{2}t}(A\cos 0.38\pi t + B\sin 0.38\pi t)$ + 45.71

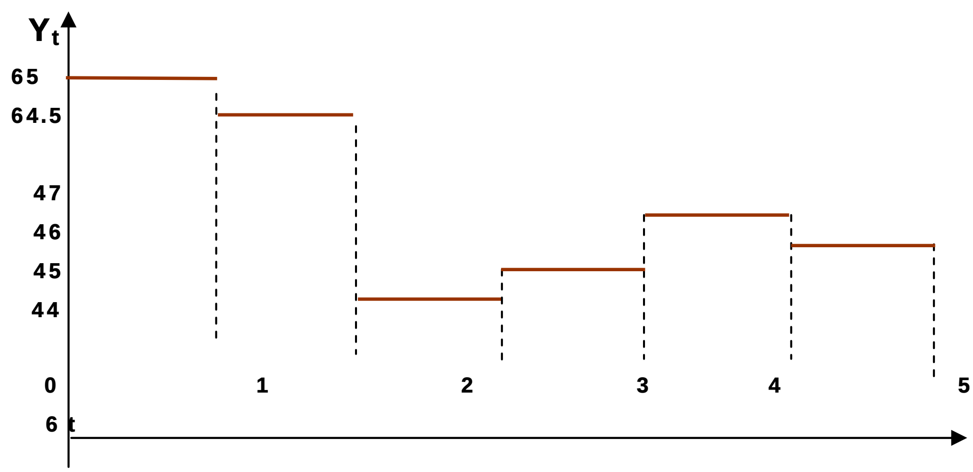
$Y_0 = 65 \Rightarrow 65 = 2^0(A\cos 0 + B\sin 0) + 45.71 \Rightarrow A = 20.71 \approx 21$

$Y_1 = 64.5 \Rightarrow 64.5 = 2(A\cos 0.38\pi + B\sin 0.38\pi) + 45.71 \Rightarrow B = 1.9 \approx 2$

$y_t = 2^{-\frac{3}{2}t}(21\cos 0.38\pi t + 2\sin 0.38\pi t)$ + 45.71

(d) Produce a graph of $\mathbf{Y_t}$ against **t** and comment upon the behaviour of $\mathbf{Y_t}$ as **t** increases. **(3 Marks)**

| t | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $Y_t$ | 65 | 64.5 | 44 | 45 | 47 | 46 |




**3.** The sequences $y_t$ and $x_t$ are linked by the following equations which holds for all $t \geq 1$ :

$\mathbf{y_t - y_{t-1} = 6x_{t-1}}$

$\mathbf{x_t = y_{t-1} + 2}$

Obtain a second order difference equation for $y_t$.

Find explicit expressions for $y_t$ and $x_t$ given that

$y_0 = 1$ and $x_0 = 1/6$

with $x_t = y_{t-1} + 2 \Rightarrow x_{t-1} = y_{(t-1)-1} + 2 = y_{t-2} + 2$

$y_t - y_{t-1} = 6x_{t-1} = 6(y_{t-2} + 2) \Rightarrow y_t - y_{t-1} - 6y_{t-2} = 12$

The auxiliary equation: $r^2 - r - 6 = 0 \Rightarrow (r+2)(r-3) = 0$

r = - 2; r = 3 two distinct real roots.

The complementary function : $y_c = A(-2)^t + B(3)^t$

The equation has the form : $y_t + ay_{t-1} + by_{t-2} = 0$

Where $a + b = -1 - 6 = -7 \neq -1$

For a particular solution , $y_p = \dfrac{c}{1+a+b} = \dfrac{12}{-6} = -2$

The general solution : $y_t = y_c + y_p = A(-2)^t + B(3)^t - 2$

$y_0 = 1 \Rightarrow A(-2)^0 + B(3)^0 - 2 = 1 \Rightarrow$ **A+ B = 3**

Since $x_0 = 1/6$ ;the first equation : $y_t - y_{t-1} = 6x_{t-1}$ gives

for t=1; $y_1 - y_0 = 6x_0 \Rightarrow y_1 - 1 = 6(1/6) \Rightarrow y_1 = 2$

$\Rightarrow A(-2)^1 + B(3)^1 = 2 \Rightarrow$ **-2A + 3B = 2**

Solving for A and B : A = 2 ; B = 1

The general solution : $y_t = 2(-2)^t + 3^t - 2$

We can use either of the original equations to find $x_t$ :

$x_t = y_{t-1} + 2 \Rightarrow x_t = 2(-2)^{t-1} + 3^{t-1} - 2 + 2 = 2(-2)^{t-1} + 3^{t-1}$

**(6 Marks)**

**4.** a) Briefly discuss how each of the following techniques can help in data reduction:

i) Clustering

ii) Box Plots **(4 Marks)**

b) Explain the difference between single linkage and complete linkage in cluster analysis. Under what circumstances would you use one of these techniques in preference to the other? **(4 Marks)**

**Refer to the Subject Guide.**

c) **( 7 Marks)**

Data were collected on seven soybean plants. Five characteristics were measured on each plant. The Euclidean distances between the vectors of measurements are displayed in the following table for all pairs of plants.

| | Plant | | | | | | |
|---|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
| Plant 1 | -- | 33 | 37 | 24 | 31 | 36 | 39 |
| Plant 2 | | -- | 42 | 22 | 39 | 42 | 35 |
| Plant 3 | | | -- | 41 | 45 | 30 | 42 |
| Plant 4 | | | | -- | 41 | 32 | 40 |
| Plant 5 | | | | | -- | 46 | 48 |
| Plant 6 | | | | | | -- | 34 |
| Plant 7 | | | | | | | -- |

Use the complete linkage clustering procedure to make three clusters.

**Complete linkage : min. to combine ,max. to calculate new distances.**

{1} {2} {3} {4} {5} {6} {7} → {1} {3} {5} {6} {7} {2, 4

| {1} | {3} | {5} | {6} | {7} | {2, 4} |
|---|---|---|---|---|---|
| - | 37 | 31 | 36 | 39 | 33 |
| | - | 45 | **30** | 42 | 41 |
| | | - | 46 | 48 | 41 |
| | | | - | 34 | 42 |
| | | | | - | 40 |

→ {1} {5} {7} {2, 4} {3, 6}

| {1} | {5} | {7} | {2, 4} | {3, 6} |
|---|---|---|---|---|
| - | **31** | 39 | 33 | 37 |
| | - | 48 | 41 | 46 |
| | | - | 40 | 42 |
| | | | - | 42 |

→ {7} {2, 4} {3, 6} {1, 5}

| {7} | {2, 4} | {3, 6} | {1, 5} |
|---|---|---|---|
| - | **40** | 42 | 48 |
| | - | 42 | 41 |
| | | - | 46 |

→ {2, 4, 7} {3, 6} {1, 5}

**5.** The doctor of a school has measured the height of pupils in a 5th grade class. The result (in cm) is as follows:

130 132 138 136 131 153 131 133 129 133 110 132 129 134 135
132 135 134 133 132 130 131 134 135 135 134 136 133 133 130

a- Which ones are outliers and why? **(2 Marks)**

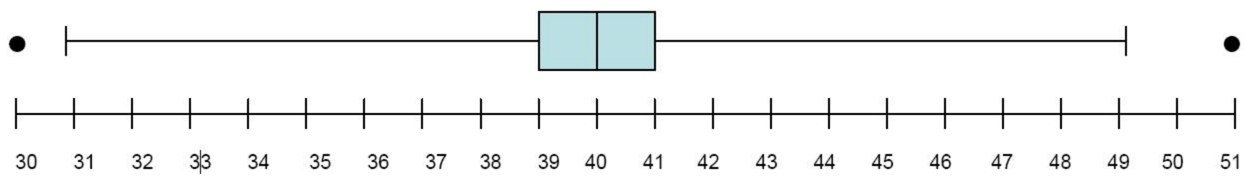
For the data set in the table the mean=132.77, s= 6.06, 3s=18.18,
z-score of the observation of 153 is (153-132.77)/6.06=3.34,
z-score of 110 is (110-132.77)/6.06=-3.76.
Since the absolute values of z-score of 153 and 110 are more than 3, the height of 153 cm and the height of 110 cm are outliers in the data set.

b- The weight of those pupils was measured in kg and the results is as follows:

37 40 39 40.5 42 51 41.5 39 41 30 40 42 40.5 39.5 41
40.5 37 39.5 40 41 38.5 39.5 40 41 39 40.5 40 38.5 39.5 41.5

Draw the box-plot for weight. **(4 Marks)**

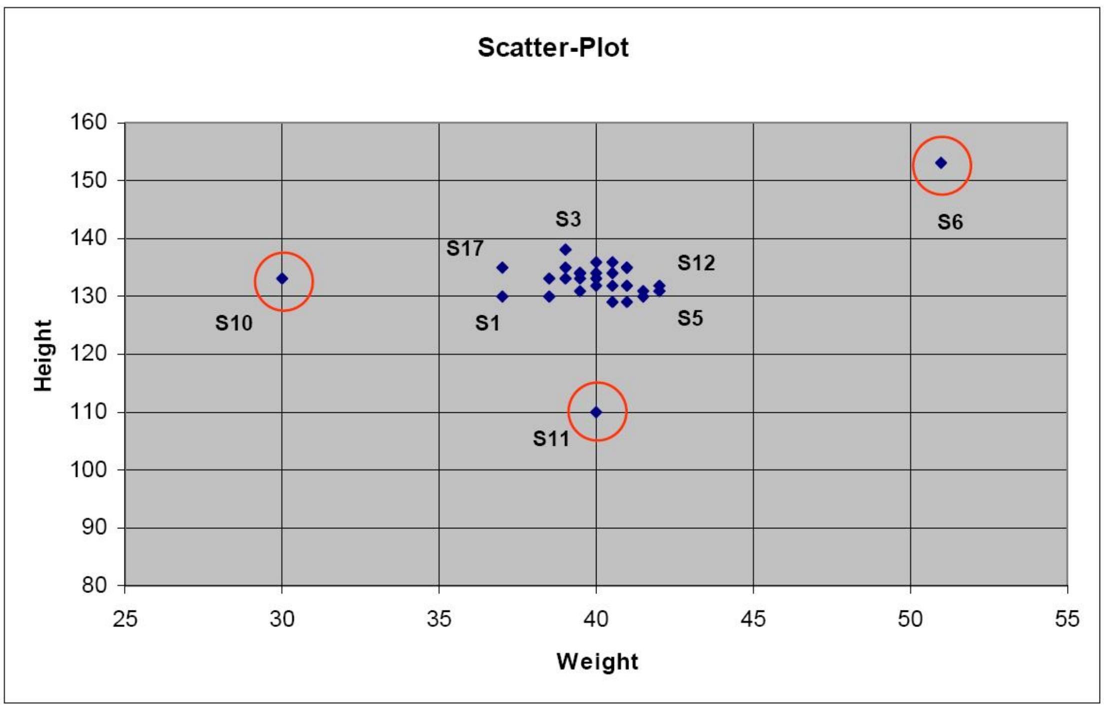
The mean for the weight is 40 kg. The Standard deviation s=3.02 and 3s=9.06. The median is 40kg
while the quartiles at 25% and 75% are 39.125 and 41 respectively. The normal distribution would range
from mean ± 3s = [30.93, 49.07]. The box-plot for weight would look like this and shows outliers 30kg and 51kg:




c- Draw the scatter-plot for both variables height and weight.

**(4 Marks)**

S6 (**H=153, W=51**); S11 (**H=110**, W=40) and S10 (H=133, **W=30**).




**<u>END of ANSWERS</u>**