International Institute for Technology and Management

Revision Problems-Tutoring Sheet #21

Unit 05a : Mathematics 1

1. Show that the graphs of the functions $f(x) = x^2 - 2x - 4$ and $g(x) = x - 8$ do not intersect, and sketch both graphs on the same diagram. Determine the positive values of the constant a such that the graph of the function $h(x) = ax - 8$ *does* intersect the graph of f.

2. A firm is a monopoly for the good it produces. Its average variable cost function is $q^2 + 4$, where q is the quantity it produces, and it has fixed costs of 20. The demand equation for its good is given by $p + q = 20$, where p is the price. Find expressions, in terms of q, for the total revenue and profit. Determine the production level q that gives maximum profit.

3. Use a matrix method to determine the numbers x, y, z satisfying

$$\begin{pmatrix} 1 & -2 & -1 \\ -2 & -3 & 5 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 2 \end{pmatrix}.$$

4. Show that the following function has one critical (or stationary) point, and determine the nature of the critical point.

$$f(x, y) = y^2 - 4xy + 4x^2 + x^2y^2.$$

5. Determine the integral $\displaystyle\int x^5 e^{x^3}\, dx$.

6. Use the Lagrange multiplier method to find the maximum value of

$$\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-1/2}$$

among all positive x, y satisfying $x + y = \sqrt{2}$

7. A geometric progression has a sum to infinity of 8, and its second term is -6. Determine all possible values for the common ratio and first term of the series.

8.(a) Find the critical (or stationary) points of the function $f(x) = x^3 e^{-x}$. Determine the nature of each critical point.

(b) The function $f(x)$, defined for $x > 0$, takes the form

$$f(x) = ax^2 + bx + \frac{c}{x},$$

for some constants a, b, c. The following facts about f and its derivative f' hold:

$$f(1) = -5, \quad f'(1) = -1, \quad \int_1^2 f(x)\,dx = \ln 2 - 4$$

Using this information, show that the following system of equations holds for a, b and c:

$$\begin{aligned} a + b + c &= -5 \\ 2a + b - c &= -1 \\ 14a + 9b + (6\ln 2)c &= 6\ln 2 - 24. \end{aligned}$$

Solve this system of equations using a matrix method, to determine a, b and c

9.(a) Using matrix methods, throughout, show that there is just one value of k for which the following system of linear equations has more than one solution, and determine all the solutions when k takes this value. What is the solution for other values of k?

$$\begin{aligned} x - y + z &= 2 \\ 3x + y - z &= 2 \\ -2x + ky + 6z &= 4. \end{aligned}$$

10.(a) The number of fleas in a carpet is 10000 at the start of 2007. Each year, 5% of the fleas die and 2000 are born. Find an expression, in as simple a form as possible, for the number of fleas N years after the start of 2007. What happens to the number of fleas in the long run?

(b) The function $f(x, y)$ is defined for $x, y > 0$ by

$$f(x,y) = \frac{ye^{2y}}{x^a},$$

where a is a fixed real number Find expressions for the partial derivatives

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial^2 f}{\partial x^2}, \ \frac{\partial^2 f}{\partial y^2}$$

Determine the values of a for which the function will satisfy the equation

$$yx^2 \frac{\partial^2 f}{\partial x^2} - 3y \frac{\partial^2 f}{\partial y^2} + 12f = 0.$$

11.(a) The supply and demand equations for a good are given, respectively, by

$$q = p^2 + 14p - 4, \quad q = -p^2 + 2p + 50,$$

where p is the price.

Determine the equilibrium price and quantity. (4 marks)

Sketch the supply and demand functions for $p \geq 0$. (4 marks)

(b) For the function

$$f(x,y) = \frac{x^2}{x+y} + e^{x/y}\sqrt{x^2 + y^2}.$$

Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y).$$

12. (a) A consumer has utility function

$$u(x,y) = \sqrt{x} + \sqrt{y}$$

for two goods, X and Y. (Here, x and y are, respectively, the amounts of X and Y consumed.) Suppose that each unit of X costs \$1 and each unit of Y costs \$2, and that the consumer has a budget of M to spend on these two goods.

By using the Lagrange multiplier method, determine the quantities x^* and y^* of X and Y that maximise the consumer's utility function subject to the constraint on his budget. (8 marks)

What is the corresponding Lagrange multiplier, λ^*? (1 marks)

If $V = u(x^*, y^*)$ is the maximum achievable utility, what is the marginal utility of income, $\dfrac{\partial V}{\partial M}$? (2 marks)

13.(a) Use the Lagrange multiplier method to find the minimum value of

$$f(x,y) = \left(\frac{1}{x}+1\right)\left(\frac{1}{y}+1\right)$$

among all positive x and y satisfying $x+y=1$. **(10 marks)**

(b) If

$$f(x,y) = x^{y^2},$$

find the partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}.$$

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1 Show that the function

$$f(x,y) = \frac{x^3}{x+y} + xy\cos\left(\frac{x}{y}\right)$$

is homogeneous of degree 2, and verify that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f(x,y).$$

2 Show that the function

$$f(x,y) = x\sin\left(\frac{x}{y}\right) + xe^{-y/x}$$

(defined for positive x and y) is homogeneous and verify that Euler's equation holds.

3 Find and classify the stationary points of the function

$f(x,y) = x^2 - 2x - y^3 + y^2 + 8$

4 Suppose that $F(x,y) = xy\sqrt{(2xy+y^2)}$. Show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 3F(x,y).$$

5. A warehouse initially contains 100 tonnes of grain. During each year, mice eat 5% of the amount of grain that was in the warehouse at the start of the year, and 3 new tonnes of grain are added to the warehouse. By solving a first-order difference equation, find a formula, in terms of N, for the amount of grain (in tonnes) in the warehouse after N years. What happens to the amount of grain in the long run?

6 A firm has production function given by

$$q(k,l) = k^{1/4}l^{1/4},$$

where k and l denote, respectively, capital and labour. Each unit of capital costs \$1 and each unit of labour costs \$16. Suppose that, when producing any given amount, the firm minimises its total expenditure on capital and labour. Show that when the production level is q, this minimum total expenditure on capital and labour is $8q^2$. **(10 marks)**

Now suppose, additionally, that the firm is a monopoly for the good it produces, and that the demand function for this good is

$$q = q^D(p) = 38 - p.$$

Suppose also that it costs the firm \$2 in raw materials for each unit of product it produces. How many units q should the firm produce in order to maximise its profit? **(10 marks)**

7 (a) Find, using the product rule, the first-order partial derivatives, of the function

$$f(x,y) = (x^2 + y^2 - 2)(xy + 7).$$

Hence determine the stationary points of the function.(You will find it helpful to consider the difference of the two first-order partial derivatives.) Classify all the stationary points.

(b) Use the Lagrange Multiplier Method to maximize

$$2\sqrt{x} + 6\sqrt{y} - z$$

subject to

$$x + y = c + z,$$

for $x, y, z \geq 0$, where c is a *positive* constant and $c < 10$.

8.(a) Three goods are sold in the same market. If their prices are x_1, x_2, x_3, then the demand quantities y_1, y_2, y_3, and the supply quantities z_1, z_2, z_3 are given by the following equations:

$$
\begin{aligned}
y_1 &= 2x_1 - x_2 - 2x_3 + 264 \\
z_1 &= 4x_1 + 4x_2 + 5x_3 - 30 \\
y_2 &= 50 - 2x_1 + x_2 + x_3 \\
z_2 &= 2x_2 + x_3 - 20 \\
y_3 &= 4x_1 + 2x_2 - 2x_3 + 4 \\
z_3 &= 16x_1 + 2x_3 - 40.
\end{aligned}
$$

Non-negative numbers x_1^*, x_2^*, x_3^* are said to be equilibrium prices if, when the prices are $x_1 = x_1^*, x_2 = x_2^*$ and $x_3 = x_3^*$, then the supply and demand quantities for each good are equal; that is, $y_1 = z_1$, $y_2 = z_2$, and $y_3 = z_3$. **Using matrix methods**, find the equilibrium prices.

9.(a) Using a matrix method express the general solution of the following system of linear equations in the form $\mathbf{v} + t\mathbf{u}$ where $\mathbf{v}$ and $\mathbf{u}$ are vectors:

$$
\begin{aligned}
x_1 + 3x_2 + 5x_3 &= 7 \\
2x_1 + 4x_2 + 6x_3 &= 8 \\
4x_1 + 11x_2 + 18x_3 &= c
\end{aligned}
$$

when $c = 25$. What happens to your solution system when $c = 0$?

10.(a) An employee retires from work and invests a lump sum of \$$L$ in a bank account that pays interest at the end of each year at a rate of 5%. The employee wants to be able to withdraw an amount of \$$D$ at the end of each of the next N years. Find an expression, in as simple a form as possible, for the minimum lump sum L that the employee will have to invest in order to make these withdrawals possible. Justify your answer fully, showing all steps in your reasoning and calculations.

(b) A consumer has utility function

$$u(x, y) = \left(x^\beta + 3y^\beta\right)^{1/\beta}$$

for two goods, X and Y, where $0 < \beta < 1$. Here, x denotes units of X and y denotes units of Y. Each unit of X costs \$1 and each unit of Y costs \$1. Find expressions, in terms of β and M, and in as simple a form as possible, for the quantities of X and Y that maximise the consumer's utility function if she spends no more than an amount \$$M$ on the two goods. Find also an expression for the maximum value of the utility function in this case.

11.(a) A firm is a monopoly for the good it produces. It has average variable cost function $AVC = q^2 + 4$ and it has fixed costs of 10. The demand equation for its good is given by $p + q = 20$, where p is the price.

Find expressions, in terms of q, for the total revenue and profit, and determine the production level q that maximises the profit. **(8 marks)**

(b) The quantity of a commodity supplied to the market when the selling price is P is believed to take the form

$$Q = aP^2 + bP + \frac{c}{P}$$

for some constants a, b, c, for $P \geq 1$. It is known that when $P = 1$ the quantity supplied is $Q = 2$; when $P = 2$ the quantity supplied is $Q = 19/2$; when $P = 3$ the quantity supplied is $Q = 62/3$. Find a system of three linear equations in the unknowns a, b, c. By using a matrix method to solve this system, find the constants a, b, c. If the formula is to be believed, what would be the quantity supplied if the price was $P = 4$? **(12 marks)**

12. (a) A consumer has utility function

$$u(x, y) = 3\sqrt{x} + \sqrt{y}$$

for two goods, X and Y. (Here, x and y are, respectively, the amounts of X and Y consumed.) Suppose that each unit of X costs \$$p$ and each unit of Y costs \$$q$, and that the consumer has a budget of B to spend on these two goods. By using the Lagrange multiplier method, determine the quantities x^* and y^* of X and Y that maximise the consumer's utility function subject to the constraint on his budget. (10 marks)

(b) Consider the following system of equations.

$$\begin{aligned} x + y - 3z &= 4 \\ 2x - y + z &= 3 \\ x + 4y + az &= b. \end{aligned}$$

Use matrix methods to determine what values a and b must take if this system is consistent and has infinitely many solutions.

What must the value of a *not* be if the system has precisely one solution?

What can be said about a and b if the system has no solutions?

13. (a) A loan of \$$L$ is taken out. The interest rate is fixed at $100r\%$ per annum and payments of \$$P$ are made at the end each year. Let y_t be the amount of loan outstanding after t repayments of P have been made.

Explain why $y_t = (1+r)y_{t-1} - P, \;\; y_0 = L.$ (2 marks)

By solving this difference equation, find y_t. (5 marks)

Suppose that the loan is to be completely repaid after N payments. Find an expression for P, in terms of r, L and N. (4 marks)

(b) For the function

$$f(x,y) = \frac{x^2}{x+y} + \frac{x}{y}\sqrt{x^2+y^2},$$

show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y).$$

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1 Show that the function

$$f(x,\,y) = \frac{x^2}{x+y} + x\sin\left(\frac{x}{y}\right)$$

is homogeneous of degree 1, and verify that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y).$$

2. A sequence of numbers $x_0, x_1, x_2, \ldots$ is given by $x_0 = 1$ and, for $n \geq 1$, $x_n + x_{n+1} = 5$. Find an explicit formula for x_n. (2 marks)

Describe in words the behaviour of the numbers x_n. (1 mark)

3. Express the following system of equations in matrix form and use a matrix method to solve it.

$$\begin{aligned} x + y + z &= 4 \\ 3x - 2y + z &= 3 \\ x - 2y + 3z &= 5 \end{aligned}$$

4. If

$$f(x,y) = y\ln\left(\frac{y}{x}\right) + xe^{x/y}$$

(defined for positive x and y), find the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f.$$

5. On the first day of 2004 in the Republic of Utopia there are 2000 on-line book retailers. During each subsequent year, the number of new such retailers grows by 60, but by the end of the year 2% of all the on-line book retailers that were in business at the start of the year will have closed down. By solving a first-order difference equation, find an expression, in terms of N, for the number of on-line book retailers N years after the first of January 2004. What happens to the number of such retailers in the long run?

6. The function $f(x, y)$ is given, for $x, y \neq 0$, by

$$f(x, y) = x + \frac{4}{y} - \frac{2y}{x}.$$

Show that f has one critical (or stationary) point and determine what type of critical point this is.

7. Find the values of x and y that will maximise the function $(x+1)^3(y+1)^2$ subject to the constraint $x + y = 13$.

8.

Determine the following integrals.

(i) $\int x^2\sqrt{x+2}\,dx.$ *[5 marks]*

(ii) $\int \frac{x}{x^2+4x+3}\,dx.$ *[4 marks]*

(iii) $\int \frac{1}{x(\ln x)^3}\,dx.$ *[5 marks]*

9.

Express the following system of equations in matrix form. Then, using a matrix method, show that there is exactly one value of c for which the system has infinitely many solutions. Find all the solutions in this case.

$$\begin{aligned} 2x + y - 3z &= 2 \\ x - y + 2z &= 2 \\ 3x + 3y + cz &= 2. \end{aligned}$$

What are the solutions for other values of c? *[10 marks]*

Find the values of x and y that minimise the function

$$f(x, y) = 8x^2 + 8xy + 12x + 10y^2 + 10y + 20$$

and verify that these values do indeed give a minimum. *[10 marks]*

10.

The function $f(x)$ is given by

$$f(x) = \frac{a}{(1+x)^2} + bx^2 + cx,$$

for some constants a, b, c. The value $f(1)$ is 13, and the derivative of f when $x = 1$ is 19. Furthermore, the area enclosed by the graph of $f(x)$, the x-axis, the y-axis, and the line $x = 1$ is $20/3$. Use these facts to show that the following system of equations must be satisfied by a, b, c:

$$a + 4b + 4c = 52,$$

$$-a + 8b + 4c = 76,$$

$$3a + 2b + 3c = 40.$$

(6 marks)

By solving this system using a *matrix method*, determine the function f. **(6 marks)**

11.

A consumer has utility function $u(x_1, x_2) = x_1 x_2^2$ for two goods, X_1 and X_2. (Here, x_1 and x_2 are, respectively, the amounts of X_1 and X_2 consumed.) Suppose that each unit of X_1 costs $\$p_1$ and each unit of X_2 costs $\$p_2$, and that the consumer has an amount $\$M$ to spend on X_1 and X_2. By using the Lagrange multiplier method, find expressions for the quantities x_1^* and x_2^* that maximise the utility function subject to the budget constraint. *[6 marks]*

What is the corresponding Lagrange multiplier, λ^*? *[2 marks]*

If $V = u(x_1^*, x_2^*)$ is the maximum achievable utility, what is the marginal utility of income, $\dfrac{\partial V}{\partial M}$? *[2 marks]*

12. (a) A firm is the only producer of two goods, X and Y. The demand equations for X and Y are given by

$$x = 50 - \frac{1}{2}p_X, \quad y = 240 - 2p_Y,$$

where x and y are the quantities of X and Y demanded (respectively) and p_X, p_Y are (respectively) the prices of X and Y. The firm's joint total cost function (that is, the cost of producing x of X and y of Y) is

$$x^2 + 2xy + y^2 + 10.$$

Find an expression in terms of x and y for the profit function. Determine the quantities x and y that maximise the profit. **(8 marks)**

(b) A consumer has utility function

$$u(x, y) = 3 \ln x_1 + \ln x_2$$

for two goods, X_1 and X_2. (Here, x_1 and x_2 are, respectively, the amounts of X_1 and X_2 consumed.) Suppose that each unit of X_1 costs $\$p_1$ and each unit of X_2 costs $\$p_2$, and that the consumer has a budget of M to spend on these two goods. By using the Lagrange multiplier method, determine the quantities x_1^* and x_2^* of X_1 and X_2 that maximise the consumer's utility function subject to the constraint on his budget.

(12 marks)

13. (a) Use the Lagrange multiplier method to find the maximum value of

$$f(x, y) = \frac{x}{(1+x)} \frac{y}{(1+y)}$$

among all positive x and y satisfying $x + y = 1$. **(10 marks)**

(b) A sequence of numbers $x_0, x_1, x_2, \ldots$ is given by $x_0 = 1$ and, for $n \geq 1$, $x_n + 3x_{n+1} = 3$. Find an explicit formula for x_n **(4 marks)**

(c) If

$$f(x, y) = x^{\sqrt{y}},$$

find the partial derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

1. Suppose that $f(x) = 2x - \ln x$ (defined for $x > 0$). Find the minimum value of $f(x)$ for $x > 0$. Use your result to show that for all positive numbers x,

$$\ln x \leq 2x - \ln 2 - 1.$$

(4 marks)

2. The supply and demand functions for a good are given, respectively, by

$$q^S(p) = 2p^2 - 8p + 39, \quad q^D(p) = 48 - 2p - p^2,$$

where p is the price. Find the equilibrium price and quantity.

(3 marks)

3. Express the following system of equations in matrix form and use a matrix method to solve it.

$$\begin{aligned} 3x + y - 2z &= 3 \\ x + y + z &= 1 \\ x + 3y - 2z &= 5 \end{aligned}$$

4. Determine the values of x and y that minimise the function

$$f(x, y) = 4x^2 + 2y^2 - 4xy - 2y + 4x - 3$$

and verify that these values of x and y do indeed minimise the function.

(4 marks)

5. Determine the integral $\displaystyle\int \frac{x}{x^2 - 4x + 3}\,dx.$ **(4 marks)**

6.

Determine the following integrals.

(i) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx.$

(ii) $\displaystyle\int \frac{x+3}{x^2 + 3x + 2}\,dx.$

(iii) $\displaystyle\int x^2 \ln x\,dx.$

7.(a) Find the critical point of the function

$$f(x, y) = \ln\left(x^2 - 2xy + 2y^2 - 2y + 5\right)$$

and show that this critical point is a local minimum.

8.

A house-buyer takes out a mortgage of amount $\$M$ with a bank. The bank's interest rate is fixed at 5% per annum and payments of $\$P$ are made at the end of each year. Let y_t be the amount of loan outstanding after t repayments of P have been made.

Explain why $y_t = (1.05)y_{t-1} - P,\ \ y_0 = M.$ *[1 mark]*

By solving this difference equation, find y_t. *[4 marks]*

Suppose that the loan is to be completely repaid after 200 payments. Find an expression for P, in terms M. *[3 marks]*

9.

) Determine the following integrals.

(i) $\int x^2 e^x \, dx$

(ii) $\int x\sqrt{3 + x}\, dx$

10.

Consider the following system of equations, where c is a constant.

$$\begin{aligned} x + 2y + z &= 4 \\ 2x - 2y - z &= 0 \\ cx + z &= 4 \end{aligned}$$

Express this system in matrix form. Using a matrix method, show that this system has solutions for all values of the constant c. For any fixed value of c, find the solutions in terms of c. **(8 marks)**

11.

A student takes out a £2,000 loan to purchase a second-hand car. He is being charged an annual interest rate of 12% on the loan which is to be repaid in equal monthly installments of £b. Let y_x be the loan balance outstanding in the x^{th} month.

(a) Show that y_x satisfies the difference equation

$$y_{x+1} = 1.01y_x - b \qquad x \in \mathbb{N}$$

(b) Solve the equation in terms of b.

(c) Find b if the loan must be repaid in three years.

12. (a) Determine the following integrals.

(i) $\displaystyle\int \frac{x+1}{x^2+7x+10}\,dx.$ **(5 marks)**

(ii) $\displaystyle\int x^2 \ln x\,dx.$ **(5 marks)**

(b) Three goods are sold in the same market. If their prices are p_1, p_2, p_3, then the demand quantities q_1^D, q_2^D, q_3^D and the supply quantities q_1^S, q_2^S, q_3^S are given by the following equations.

$$\begin{aligned}
q_1^D &= 40 - p_1 + p_2 + 2p_3 \\
q_1^S &= p_1 + 2p_2 + 7p_3 - 20 \\
q_2^D &= 20 + p_1 - p_2 + p_3 \\
q_2^S &= 3p_1 + 2p_2 - 120 \\
q_3^D &= 20 + p_1 + p_2 - p_3 \\
q_3^S &= p_1 + 2p_2 - 30.
\end{aligned}$$

The equilibrium prices are the non-negative numbers p_1^*, p_2^*, p_3^* with the property that when the prices are $p_1 = p_1^*, p_2 = p_2^*$ and $p_3 = p_3^*$, then the supply and demand quantities for each good are equal. **Using matrix methods**, find p_1^*, p_2^*, p_3^*. **(10 marks)**

13.

(b) Determine the following integrals.

(i) $\int x(x+2)^{3/2}\,dx.$ (4 marks)

(ii) $\int \frac{x+1}{x^2+2x+2}\,dx.$ (3 marks)

(c) A geometric series $x_0, x_1, x_2, \ldots$ takes the form $x_t = ar^{t-1}$. It has a sum to infinity of 3, and $x_1 = 2/3$. Show that there are two possible values of r, and determine the corresponding values of a. (5 marks)

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1 (a) A firm has fixed costs of 10 and its marginal revenue and marginal cost functions are given, respectively, by

$$MR = 11 - q, \quad MC = q^2 - 3q + 3,$$

where q is the level of production. Determine the value of q which maximises profit, and determine also the value of this maximal profit. *[6 marks]*

(b) A firm is the only producer of two goods, X and Y. The prices p_X and p_Y of X and Y are related to the quantities, x and y, produced, as follows:

$$p_X + 4x = 100, \quad p_Y + 2y = 60.$$

The firm's joint total cost function (that is, the cost of producing x of X and y of Y) is

$$2x^2 + 4xy + 2y^2 + 40.$$

Find an expression in terms of x and y for the profit function. Determine the quantities x and y that maximise the profit, and find the corresponding prices p_X, p_Y. *[10 marks]*

2.

(b) Determine the following integrals.

(i) $\int \frac{\ln x}{x^2}\,dx$ **(6 marks)**

(ii) $\int \frac{\sin x}{(\cos x)^2 + 6\cos x + 8}\,dx$ **(6 marks)**

4. If

$$f(x, y) = \ln\left(\frac{y}{x}\right)(x^2 + y^2)^{3/2}$$

(defined for positive x and y), find the partial derivatives

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

and show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f.$$

4 Using a matrix method, find all the solutions to the following system of equations:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 4 \\ x_1 - 3x_2 + 2x_3 &= 4 \\ 3x_1 + x_2 &= 4. \end{aligned}$$

4 Using a matrix method, find all the solutions to the following system of equations:

$$\begin{aligned} 2x - 3y + z &= -1 \\ x - y - z &= 0 \\ 2x - 5y + 7z &= -3. \end{aligned}$$

4. Determine the values of x and y that minimise the function

$$f(x, y) = 5x^2 + 2y^2 + 2xy - 6x - 6y + 5$$

and verify that these values of x and y do indeed minimise the function. (5 marks)

5. A firm has marginal cost function Qe^{Q^2} and its fixed costs are 5. Find its total cost function. (4 marks)

7.(a) Find and classify the three stationary points of the following function:

$$f(x,y) = 6x^3 + 9x^2y - 9x^2 + y^3 + 3y^2 - 9y.$$

[To classify the point (0,1), you will find it helpful to consider the behaviour of $f(x,1)$ near $x = 0$.]

(b) The function $f(x)$ is of the form

$$f(x) = \frac{a}{x} + bx + cx^2,$$

for some numbers a, b and c. When $x = 1$, $f(x) = 4$, when $x = 2$, $f(x) = 4$, and when $x = 1$, the derivative $f'(x)$ equals -3.

Show that the following system of linear equations holds for a, b and c:

$$\begin{aligned} a + b + c &= 4 \\ a + 4b + 8c &= 8 \\ a - b - 2c &= 3. \end{aligned}$$

4. Determine the values of x and y that minimise the function

$$f(x,y) = 5x^2 + 2y^2 + 2xy + 2x + 4y + 2$$

and verify that these values of x and y do indeed minimise the function.
(5 marks)

(b) Determine the following integrals.

(i) $\int x^2 e^{2x}\, dx.$ **(4 marks)**

(ii) $\int x^3 \sqrt{x^2+1}\, dx.$ **(4 marks)**

(c) A sequence of numbers $x_0, x_1, x_2, \ldots$ is given by $x_0 = 2$ and, for $n \geq 1$, $x_n - 2x_{n+1} = 3$. Find an explicit formula for x_n **(4 marks)**

7.(a) Find the critical point of the function

$$f(x,y) = 2^{(x^2 - 2xy + 2y^2 - 2y + 2)}$$

and show that this critical point is a local minimum.

(b) A firm's production function $P : \mathbf{R}^2 \to \mathbf{R}$ is given by

$$P(x, y) = 100(\frac{1}{5}x^{\frac{1}{2}} + \frac{4}{5}y^{\frac{1}{2}})^2$$

where x and y represent quantities of two goods. Use Lagrange multipliers to find production levels which maximize production under the budget constraint

$$5x + 2y \leq 2050.$$

(b) A firm is the only producer of two goods, X and Y. The demand equations for X and Y are given by

$$x = 100 - p_X, \quad y = 200 - 2p_Y,$$

where x and y are the quantities of X and Y demanded (respectively) and p_X, p_Y are (respectively) the prices of X and Y. The firm's joint total cost function (that is, the cost of producing x of X and y of Y) is

$$x^2 + 2xy + y^2 + 10.$$

Find an expression in terms of x and y for the profit function. Determine the quantities x and y that maximise the profit.

1 Determine the following integrals:

$$\int \frac{2\sin x \cos x}{1 + \sin^2 x} dx,$$

$$\int (\ln(x+1))^2 \, dx,$$

$$\int \frac{x^5}{\sqrt{2x^2+3}} \, dx.$$

1 Determine the following integrals:

$$\int \frac{1}{\sqrt{2x - x^2}} dx,$$

$$\int (\ln x)^2 \, dx,$$

$$\int \frac{x^5}{\sqrt{2x^2 + 3}} \, dx.$$

In the first integral you may wish to complete the square.

1. Determine the following integrals.

$$\int \frac{\ln x}{\sqrt{x}} \, dx,$$

$$\int \frac{x}{4x^2 - 1} \, dx,$$

$$\int \frac{x + 2}{x^2 + 2x + 4} \, dx.$$

1. Determine the following integrals. In the last integral write the numerator as $x \cdot 2x$ and use integration by parts.

$$\int \frac{\cos x}{\sqrt{1 + \sin x}} \, dx,$$

$$\int \frac{\sqrt{x - 1}}{x} \, dx,$$

$$\int \frac{2x^2}{(1 + x^2)^2} \, dx.$$

1. Determine the following integrals.

$$\int \frac{\ln x}{x^{3/2}}\, dx,$$

$$\int \frac{x}{1-x^2}\, dx,$$

$$\int \frac{x+1}{x^2+x+2}\, dx.$$

11. (a) The supply and demand equations for a good are given, respectively, by

$$q = p^2 + 4p - 1, \quad q = -p^2 - 6p + 27,$$

where p is the price.

Determine the equilibrium price and quantity. (4 marks)

Sketch the supply and demand functions for $p \geq 0$. (4 marks)

(b) Determine the following integrals.

(i) $\int x^2\sqrt{x+2}\, dx.$ (4 marks)

(ii) $\int \frac{x}{x^2+3x+2}\, dx.$ (4 marks)

(b) Determine the following integrals:

$$\int_0^{\pi/4} \frac{1}{\sqrt{\tan x}\, \cos^2 x}\, dx,$$

$$\int (x+2)^2 \ln x\, dx$$

(b) The function $f(x)$ is of the form

$$f(x) = \frac{a}{x^2} + \frac{b}{x} + cx,$$

for some numbers a, b and c. When $x = 1$, $f(x) = 9$, when $x = 2$, $f(x) = 5$, and when $x = 1$, the derivative $f'(x)$ equals -11.

Show that the following system of linear equations holds for a, b and c:

$$\begin{aligned} a + b + c &= 9 \\ a + 2b + 8c &= 20 \\ 2a + b - c &= 11. \end{aligned}$$

(4 marks)

Solve this system using a matrix method, to find the numbers a, b, c.

(6 marks)

Two functions $V(x, y)$ and $U(x, y)$ are connected by the equation

$$V(x, y) = U(x, y)e^{-ax-by}$$

where a and b are constants. Find

$$\frac{\partial V}{\partial x}, \ \frac{\partial V}{\partial y}, \ \frac{\partial^2 V}{\partial x^2}$$

in terms of U and its partial derivatives.

Suppose that V satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial^2 V}{\partial x^2} + 2\frac{\partial V}{\partial x} - 3V.$$

Let

$$a = 1, \quad b = 4.$$

Show that the function U then satisfies the equation

$$\frac{\partial U}{\partial y} = \frac{\partial^2 U}{\partial x^2}.$$