International Institute for
Technology and Management

Tutoring Sheet # 16

Unit 05a : Mathematics 1

Solution

1. Suppose that $f(x,y) = x^2y$. Let $x = 2 - t$ and $y = 3t + 7$
Use the chain rule to find F′(t) .

$$F'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2xy(-1) + x^2(3) = -2(2-t)(3t+7) + 3(2-t)^2$$

2. Suppose that $f(x,y) = x^2y + y^2$ .Let $x = 3t^2 + 3$ and $y = t^3 - 7$
Use the chain rule to find F′(2)

$$F'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2xy(6t) + (x^2+2y)(3t^2)$$

$$F'(t) = 2(3t^2 + 3)(t^3 - 7)(6t) + [(3t^2 + 3)^2 + 2(t^3 - 7)](3t^2)$$

3. Find dy/dx in each of the following cases:

a.) $2x^2 + 5xy + y^2 = 19$ $\quad dy/dx = -\dfrac{\partial g/\partial x}{\partial g/\partial y} = -\dfrac{4x+5y}{5x+2y}$

b.) $x^4y^3 + 4x^2y^2 - 2x^5y = 3$ $\quad dy/dx = -\dfrac{4x^3y^3 + 8xy^2 - 10x^4y}{3x^4y^2 + 8x^2y - 2x^5y}$

c.) $x^2y^3 - 6x^3y^2 + 2xy = 1$ $\quad dy/dx = -\dfrac{2x\ y^3 - 18x^2y^2 + 2y}{3x^2y^2 - 12x^3y + 2x}$

d.) $e^{xy} + 2(x+y) = 5$ $\quad dy/dx = -\dfrac{ye^{xy} + 2}{xe^{xy} + 2}$

4. Find and classify the critical points of the following functions :

a.) $f(x,y) = 2x^2 + 2y^2$: To find the critical point , $f_1 = 0$; $f_2 = 0$
$f_1 = 4x = 0 \Rightarrow$ **$x = 0$** ; $f_2 = 4y = 0 \Rightarrow$ **$y = 0$** ; (0,0) critical pt.
To check whether it maximizes or minimizes f or a saddle point :
$(f_{11})(f_{22}) - f_{12}^{\,2}$
$f_{11} = 4$, $f_{22} = 4$; $f_{12} = 0 \Rightarrow (f_{11})(f_{22}) - f_{12}^{\,2} = 16 > 0$
Since $f_{11} = 4 > 0$, (0 , 0) minimizes f.

b.) f(x,y) = $e^{-(x^2+y^2)}$ = $e^{-x^2-y^2}$

f_1= $-2xe^{-x^2-y^2}$ = 0 $\Rightarrow$ x = 0 ; f_2= $-2ye^{-x^2-y^2}$ = 0 $\Rightarrow$ y = 0

(0,0) critical point.

f_{11} = $-2e^{-x^2-y^2}$ + (-2x)(-2x) $e^{-x^2-y^2}$ =($4x^2$ – 2) $e^{-x^2-y^2}$

[Using u = -2x ; v = $e^{-x^2-y^2}$,then u′v + v′u]

With x = 0 , y = 0 ; f_{11} = - $2e^0$ = - 2

f_{22} =$-2e^{-x^2-y^2}$ + (-2y)(-2y) $e^{-x^2-y^2}$ = ($4y^2$-2)$e^{-x^2-y^2}$,

with x = 0 ,y = 0 ; f_{22} = $-2e^0$ = - 2

f_{12} = (– 2x)(-2y) $e^{-x^2-y^2}$; with x = 0 , y= 0 ; f_{12} = 0

(f_{11})(f_{22}) – $f_{12}^{\ 2}$ = 4 > 0 ;Since f_{11} = -2 < 0 ,(0,0) maximizes f .

c.) f(x,y) = x^2 - y^2 Same as part (a) : (0,0) is the critical point.

f_{11} = 2 , f_{22} = 2 ; f_{12} = 0 $\Rightarrow$ (f_{11})(f_{22}) – $f_{12}^{\ 2}$ = 4 > 0

Since f_{11} = 2 > 0 , (0 , 0) minimizes f.

d.) f(x,y) = y^3 + 3xy – x^3

f_1= 3y – $3x^2$ = 0 $\Rightarrow$ **$y = x^2$** substitute this in f_2; f_2 = $3y^2$+3x=0

$3(x^2)^2$ + 3x = 0 $\Rightarrow$3(x^4 + x) = 0 $\Rightarrow$ x(x^3+1) = 0

$\Rightarrow$ x(x+1)(x^2 –x +1) = 0 $\Rightarrow$ x = 0 or x = - 1

x = 0 $\Rightarrow$ y = x^2 = 0 $\Rightarrow$ (0 , 0) first critical point

x = -1 $\Rightarrow$ y = x^2 = 1 $\Rightarrow$ (-1,1) second critical point

(0 , 0) first critical point

f_{11} = -6x = 0 , f_{22} = 6y = 0 ; f_{12} = 3 $\Rightarrow$ (f_{11})(f_{22}) – $f_{12}^{\ 2}$ =- 9< 0

Therefore (0,0) is a saddle point.

(-1,1) second critical point

f_{11} = -6x = 6 , f_{22} = 6y = 6 ; f_{12} = 3 $\Rightarrow$ (f_{11})(f_{22}) – $f_{12}^{\ 2}$ =32 > 0

Since f_{11} = 6 > 0 $\Rightarrow$ (-1,1) minimizes f .

e.) f(x,y) = 6 + 4x -$3x^2$ + 4y +2xy – $3y^2$

f_1= 4 -6x + 2y = 0

f_2= 4 + 2x -6y = 0

Solving simultaneously, x = 1 , y = 1$\Rightarrow$ (1,1) is the critical point.

f_{11} = -6 , f_{22} = -6; f_{12} = 2 $\Rightarrow$ (f_{11})(f_{22}) – $f_{12}^{\ 2}$ =32 > 0

Since f_{11} = -6 < 0 $\Rightarrow$ (1,1) maximizes f .

5. Find the critical point of the function

$f(x,y) = Ln(x^2 - 2xy + 2y^2 - 2y + 2)$

and show that this critical point is a local minimum.

$f_1 = \dfrac{2x-2y}{x^2-2xy+2y^2-2y+2} = 0 \Rightarrow 2x - 2y = 0 \Rightarrow x = y$----(1)

$f_2 = \dfrac{-2x+4y-2}{x^2-2xy+2y^2-2y+2} = 0 \Rightarrow -2x + 4y - 2 = 0$ -----(2)

Substitute y = x in (2) : $-2x + 4x - 2 = 0 \Rightarrow x = 1 \Rightarrow y = 1$

(1,1) is the critical point.

$$f_{11} = \frac{2(x^2-2xy+2y^2-2y+2)-(2x-2y)(2x-2y)}{(x^2-2xy+2y^2-2y+2)^2}$$

[Using $u = 2x - 2y$; $v = x^2-2xy+2y^2-2y+2$;then $\dfrac{u'v-v'u}{v^2}$]

With x =1 , y = 1 , $\mathbf{f_{11} = 2}$. Similarly ,

$$f_{22} = \frac{4(x^2-2xy+2y^2-2y+2)-(-2x+4y-2)(-2x+4y-2)}{(x^2-2xy+2y^2-2y+2)^2}$$

With x = 1 , y = 1 , $\mathbf{f_{22} = 4}$

$$f_{12} = \frac{-2(x^2-2xy+2y^2-2y+2)-(2x-2y)(-2x+4y-2)}{(x^2-2xy+2y^2-2y+2)^2}$$

With x = 1 , y = 1 , $\mathbf{f_{12} = -2}$

$(f_{11})(f_{22}) - f_{12}^{\ 2} = 4 > 0$

Since $f_{11} = 2 > 0 \Rightarrow$ (1,1) minimizes f .

6. Find the values of x and y that minimize the function:

$f(x,y) = 8x^2 + 10y + 8xy + 10y^2 + 12x + 6$

and verify that these values do indeed give a minimum.

$f_1 = 16x + 8y + 12 = 0$

$f_2 = 10 + 8x + 20y = 0$

Solving simultaneously, x = -5/8,y = -1/4 $\Rightarrow$ (-5/8,-1/4) is the critical point.

$f_{11} = 16$, $f_{22} = 20$; $f_{12} = 8 \Rightarrow (f_{11})(f_{22}) - f_{12}^{\ 2} = 256 > 0$

Since $f_{11} = 16 > 0 \Rightarrow$ (-5/8,-1/4) minimizes f.