International Institute for
Technology and Management

Unit 05a:Mathematics 1 **Tutoring Sheet #14**

REVISION - ANSWERS

1. The functions f(x) and g(x) are given by :

$f(x) = 4x^2 - 8x - 1$, $g(x) = -4x^2 - 2x - 1$
Sketch the graphs of $y = f(x)$ and $y = g(x)$ for $\mathbf{x > 0}$
on the same diagram, and determine the positive value of x at which these two graphs intersect.

$f(x) = 4x^2 - 8x - 1$ for $\mathbf{x > 0}$

-It should be realized that f(x) has a parabolic U shape since it has a positive x^2 term.

-**An accurate sketch will need to indicate where the curve cuts the axes:**

x-intercepts : $y = 0 \Rightarrow 4x^2 - 8x - 1 = 0$

This can be solved using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 4(4)(-1)}}{2(4)}$$

$x = \dfrac{8 \pm \sqrt{80}}{8}$ and the x-intercepts are:

$\left(\dfrac{8 + \sqrt{80}}{8}, 0\right)$ and $\left(\dfrac{8 - \sqrt{80}}{8}, 0\right)$

These values should be left like this –indeed, this has to be since no calculators can be used.

There is one thing you may do to simplify it further if you notice that $\sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$

Then $x = \dfrac{8 \pm \sqrt{80}}{8} = \dfrac{8 \pm 4\sqrt{5}}{8} = \dfrac{2 \pm \sqrt{5}}{2}$ and hence the

x-intercepts become: $\left(\dfrac{2 + \sqrt{5}}{2}, 0\right)$ and $\left(\dfrac{2 - \sqrt{5}}{2}, 0\right)$

y-intercept: $x = 0 \Rightarrow y = -1 \therefore (0, -1)$

- An accurate sketch will need to show the minimum of the graph of f(x),we know it's a minimum from the U shape.

The minimum can be found in **one of two ways** :

By differentiation : $f(x) = 4x^2 - 8x - 1 \Rightarrow f'(x) = 8x - 8 = 0$

$\Rightarrow x = 1$, substituting this in f(x) , $y = 4(1)^2 - 8(1) - 1 = -5$

$\therefore (1, -5)$

OR by finding the vertex : $x = \frac{-b}{2a} = \frac{-(-8)}{2(4)} = \frac{8}{8} = 1$

Substituting this in f(x) , $y = 4(1)^2 - 8(1) - 1 = -5 \Rightarrow V(1, -5)$

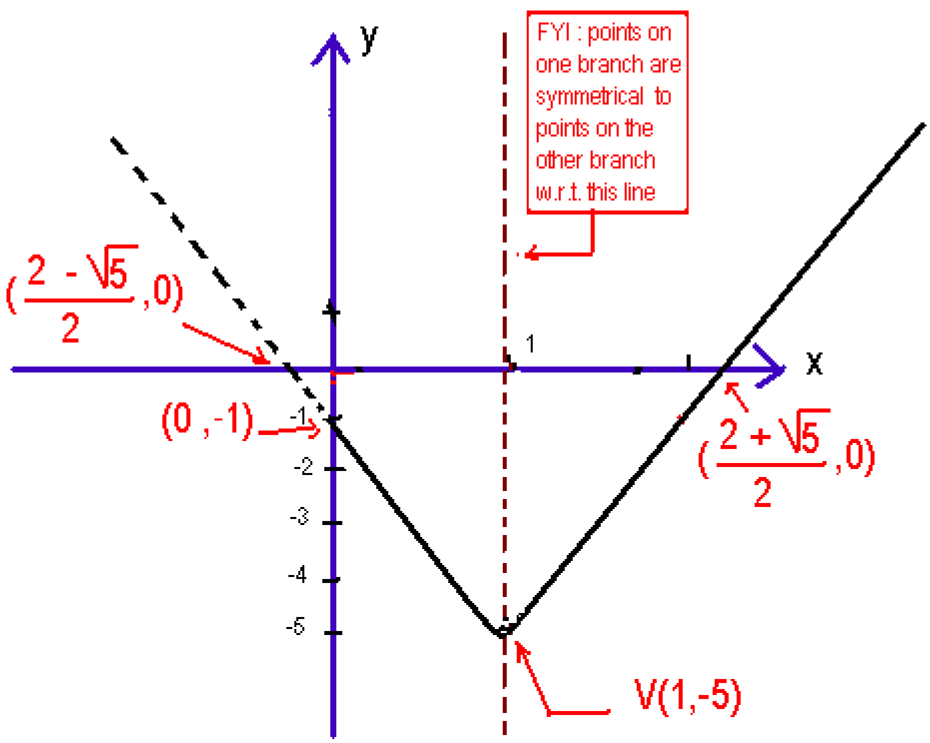
Now you can Sketch the graph of f(x):

-You know it has a U shape

- You know the intercepts with the axes:

$\left(\frac{2+\sqrt{5}}{2}, 0\right)$, $\left(\frac{2-\sqrt{5}}{2}, 0\right)$ and (0 , -1)

-You know the Vertex (minimum) : **(1, - 5)**



Since x > 0 , the dotted part is not considered

You may find it difficult to plot the graph if you choose <u>equal units</u> of length on both axes; this is why I choose the unit on the x-axis larger than that of the y-axis.

g(x) = -4x² – 2x – 1 for x > 0

-It should be realized that g(x) has a parabolic I shape since it has a negative x^2 term.

-**An accurate sketch will need to indicate where the curve cuts the axes:**

x-intercepts : y = 0 ⇒ $-4x^2 - 2x - 1 = 0 \Rightarrow 4x^2 + 2x+1=0$

This can be solved using the quadratic formula:

$b^2 - 4ac = 2^2 - 4(4)(1) = -12 < 0$

hence the equation has no real root and therefore the graph does not cut the x-axis.

y-intercept: x = 0 ⇒ y = -1 ∴ (0 , - 1)

-An accurate sketch will need to show the maximum of the graph of f(x),we know it's a maximum from the I shape.

The maximum can be found in **one of two ways** :

By differentiation : $f(x) = -4x^2 - 2x - 1 \Rightarrow f'(x) = -8x - 2 = 0$

$x = \frac{-1}{4}$, substituting this in f(x) ,$y = 4(\frac{-1}{4})^2 - 2(\frac{-1}{4}) - 1 = \frac{-1}{4}$

$\therefore (\frac{-1}{4}, \frac{-1}{4})$

OR by finding the vertex : $x = \frac{-b}{2a} = \frac{-(-2)}{2(-4)} = \frac{2}{-8} = \frac{-1}{4}$

substituting this in f(x) ,$y = 4(\frac{-1}{4})^2 - 2(\frac{-1}{4}) - 1 = \frac{-1}{4}$

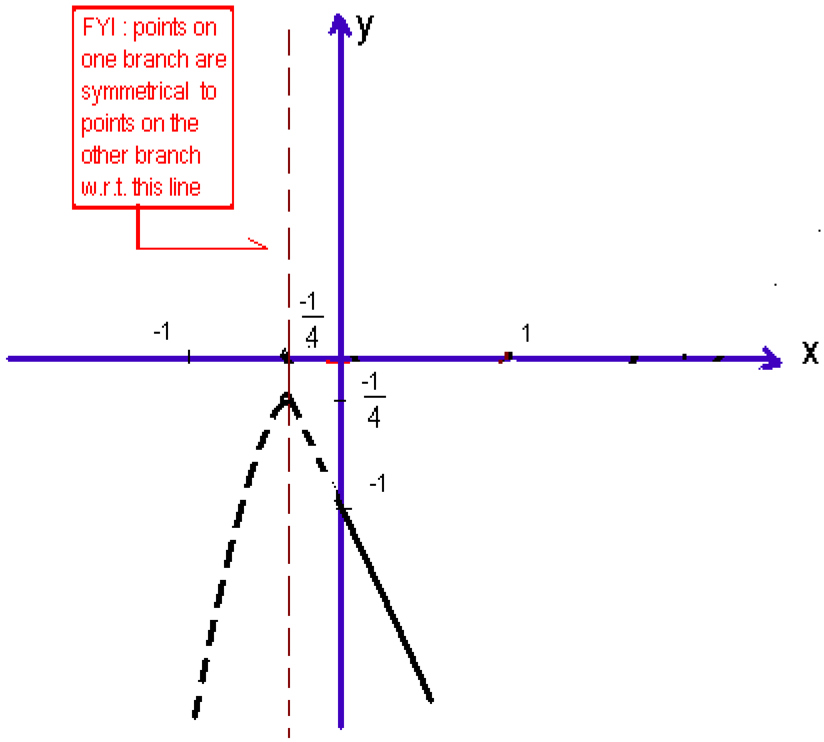
$\Rightarrow V(\frac{-1}{4}, \frac{-1}{4})$

Now you can Sketch the graph of f(x):

-You know it has a I shape

- You know the intercepts with the axes:
  no intersection with x-axis and (0 , -1)

-You know the Vertex (maximum) : $(\frac{-1}{4}, \frac{-1}{4})$



Since x > 0 , the dotted part is not considered

It is never adequate to determine a few points on the curve and then join them up, this is *plotting* not *Sketching*.

To determine the points of intersection, we solve:
$4x^2 - 8x - 1 = -4x^2 - 2x - 1 \Rightarrow 8x^2 - 6x = 0 \Rightarrow 2x(4x - 3) = 0$

either x = 0 or $x = \frac{3}{4} > 0$ which is the required.

2. The supply equation for a good is $q = p^2 + 7p - 2$ and the demand equation is $q = -p^2 - p + 40$ where p is the price. Sketch the supply and the demand functions for $p \geq 0$

The supply equation : $q = p^2 + 7p - 2$ for $p \geq 0$

The fact that q is given as a function of p suggests that it is natural to place p on the horizontal and q on the vertical axis.

(1)The supply curve has a U shape since it has a positive p^2 term

(2) Intercepts :

<u>p-intercepts</u> : $q = 0 \Rightarrow p^2 + 7p - 2 = 0$

This can be solved using the quadratic formula:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{49 - 4(1)(-2)}}{2(1)}$$

$$p = \frac{-7 \pm \sqrt{57}}{2}$$ and the p-intercepts are:

$$\left(\frac{-7 - \sqrt{57}}{2}, 0\right) \text{ and } \left(\frac{-7 + \sqrt{57}}{2}, 0\right)$$

These values should be left like this –indeed, this has to be since no calculators can be used.

<u>q-intercept</u> : $p = 0 \Rightarrow q = -2$; $(0, -2)$

(3) The minimum can be found in **one of two ways** :

By differentiation : $\frac{dq}{dp} = 2p + 7 = 0 \Rightarrow p = \frac{-7}{2}$

Substituting this in q ,$q = (\frac{-7}{2})^2 + 7(\frac{-7}{2}) - 2 = \frac{-57}{4}$

$\therefore (\frac{-7}{2}, \frac{-57}{4})$

OR by finding the vertex : $p = \frac{-b}{2a} = \frac{-7}{2(1)} = \frac{-7}{2} \Rightarrow q = \frac{-57}{4}$

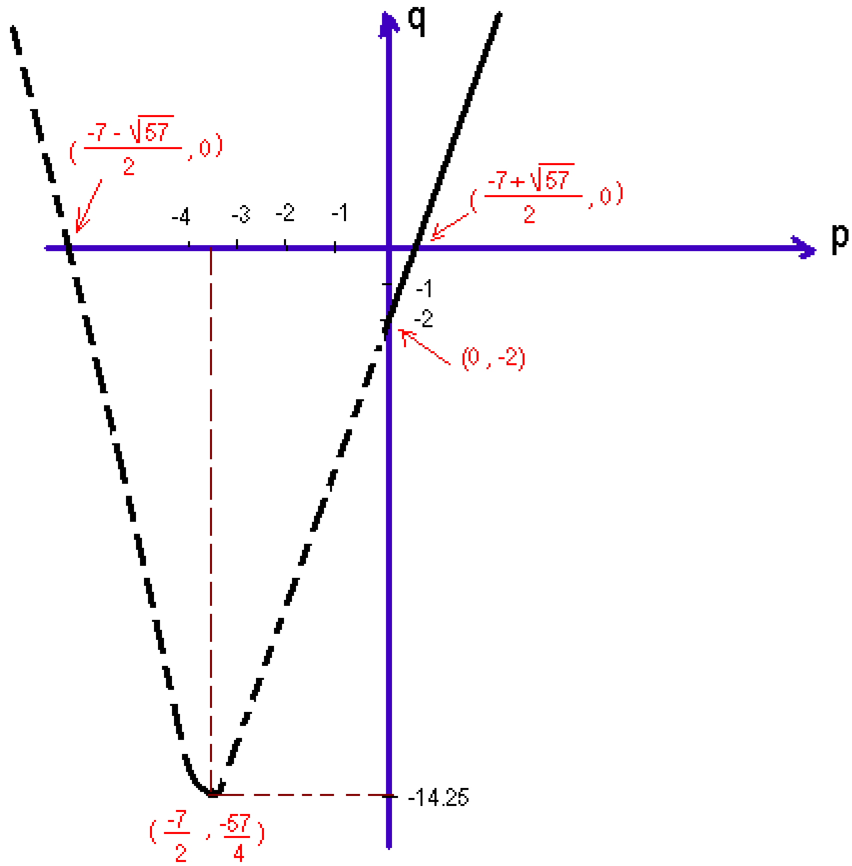
Now you can Sketch the graph of supply function:

-You know it has a U shape

- You know the intercepts with the axes:

$$\left(\frac{-7-\sqrt{57}}{2},0\right),\ \left(\frac{-7+\sqrt{57}}{2},0\right)$$ and (0 , -2)

-You know the Vertex (minimum) : $(\frac{-7}{2}, \frac{-57}{4})$



The dotted part is not considered since $p \geq 0$

The demand equation $q = -p^2 - p + 40$ for $p \geq 0$

(1) The demand curve has a I shape since it has a negative p^2 term.

(2) Intercepts :

p-intercepts : $q = 0 \Rightarrow -p^2 - p + 40 = 0$

This can be solved using the quadratic formula:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(-1)(40)}}{2(-1)}$$

$p = \dfrac{1 \pm \sqrt{161}}{-2}$ and the p-intercepts are:

$$\left(\frac{-1 - \sqrt{161}}{2}, 0\right) \text{ and } \left(\frac{-1 + \sqrt{161}}{2}, 0\right)$$

These values should be left like this –indeed, this has to be since no calculators can be used.

<u>q-intercept</u> : $p = 0 \Rightarrow q = 40$; (0 , 40)

(3) The maximum can be found in **one of two ways** :

By differentiation : $\dfrac{dq}{dp} = -2p - 1 = 0 \Rightarrow p = \dfrac{-1}{2}$

Substituting this in q ,$q = -\left(\dfrac{-1}{2}\right)^2 - \left(-\dfrac{1}{2}\right) + 40 = \dfrac{163}{4}$

$\therefore \left(\dfrac{-1}{2}, \dfrac{163}{4}\right)$

OR by finding the vertex: $p = \dfrac{-b}{2a} = \dfrac{1}{2(-1)} = \dfrac{-1}{2} \Rightarrow q = \dfrac{163}{4}$

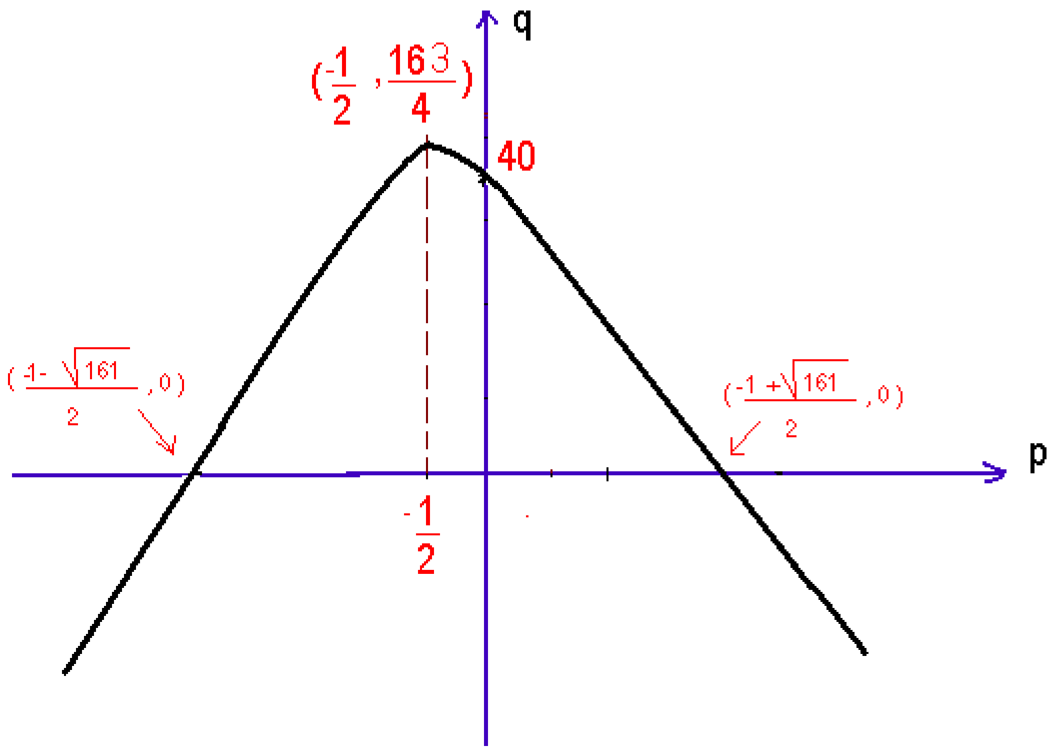
Now you can Sketch the graph of demand function:

-You know it has a I shape

- You know the intercepts with the axes:

$$\left(\frac{-1 - \sqrt{161}}{2}, 0\right), \left(\frac{-1 + \sqrt{161}}{2}, 0\right) \text{ and } (0, 40)$$

-You know the Vertex (maximum) : $\left(\dfrac{-1}{2}, \dfrac{163}{4}\right)$



Determine the equilibrium price and quantity:

We solve : $p^2 + 7p - 2 = -p^2 - p + 40$

Which is equivalent to : $2p^2 + 8p - 42 = 0 \Rightarrow p^2 + 4p - 21 = 0$

$(p - 3)(p+7) = 0$, that is $p = -7$ or $p = 3$ of which only 3 is economically meaningful .

The equilibrium quantity ,substitute $p = 3$ in any of the equations:

$q = p^2 + 7p - 2 = 3^2 + 7(3) - 2 = 28$.

3. A firm's cost function is C = 20q + 60 and the revenue is $R = q^2 - 8q$. Sketch the graphs of C and R on the same diagram .Find the break even value of q.

The fact that C and R are given as functions of q suggests that it is natural to place q on the horizontal and C & R on the vertical axis.

C = 20q + 60

Intercepts : q = 0 $\Rightarrow$ C = 60 (0 , 60)

C = 0 $\Rightarrow$ q = - 3 (-3 , 0)

$R = q^2 - 8q$

(1) The curve has a U shape since it has a positive q^2 term.

(2) Intercepts :

q-intercepts : R = 0 $\Rightarrow$ $q^2 - 8q = 0$ $\Rightarrow$ q(q-8) = 0

Either q = 0 or q = 8 and the p-intercepts are:

(0 , 0) and (8 , 0)

R-intercept : q = 0 $\Rightarrow$ R = 0 ; (0 , 0)

(3) The minimum can be found in **one of two ways** :

By differentiation : $\frac{dR}{dq}$ = 2q - 8 = 0 $\Rightarrow$ q = 4

Substituting this in R ,R = 4^2 - 8(4)= -16

$\therefore$ (4, -16)

OR by finding the vertex: $p = \frac{-b}{2a} = \frac{8}{2(1)} = 4 \Rightarrow$ q =-16

$\therefore$ V(4, -16)

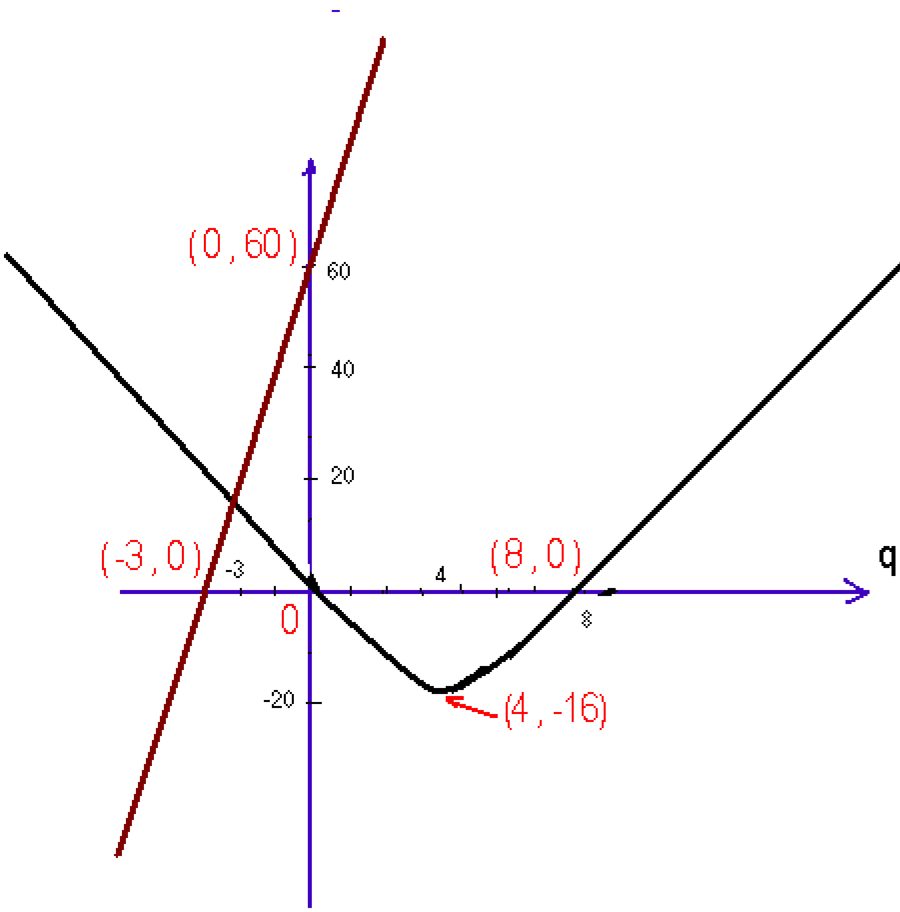
Now you can Sketch the graph of demand function:

-You know it has a U shape

- You know the intercepts with the axes:

(0 , 0) and (8 , 0)

-You know the Vertex (minimum) : (4 , -16)



To find the break even value of q: we solve R = C
$q^2 - 8q = 20q + 60 \Rightarrow q^2 - 28q - 60 = 0$
$(q + 2)(q - 30) = 0$ .Either q = - 2 or q = 30 of which only 30 is economically meaningful.

4. A monopolist's average cost function is given by :

$$9 + \frac{3}{10}q + \frac{30}{q}$$

Where q is the quantity produced, the demand function for the good is $q = 40 - \frac{4}{3}p$

Determine expressions, in terms of q , for the revenue and the profit and determine the value of q that maximizes the profit. Find the maximum profit.

Given : AC = $9+\frac{3}{10}q+\frac{30}{q}$

Demand : q = 40 - $\frac{4}{3}p$

Revenue ? Profit ?

Revenue =Demand × Price = q × p

q = 40 - $\frac{4}{3}p \Rightarrow p=\frac{3q-120}{-4}=\frac{-3q}{4}+30$

R = q× $\left(\frac{-3q}{4}+30\right)=\frac{-3q^2}{4}+30q$

Profit = Revenue – Cost

AC = $9+\frac{3}{10}q+\frac{30}{q} \Rightarrow$ TC = q ×AC = $9q+\frac{3}{10}q^2+30$

Profit: Π = $\frac{-3q^2}{4}+30q$ - $\left(9q+\frac{3}{10}q^2+30\right)$

Π = $\frac{-21}{20}q^2+21q-30$

q=? so that Π is maximum : $\frac{d\prod}{dq}=0$

$\frac{-21}{10}q+21=0 \Rightarrow$ q = 10

Maximum profit ? substitute q = 10 in the profit function

Π = $\frac{-21}{20}q^2+21q-30$ = $\frac{-21}{20}(10)^2+21(10)-30$ = 75

5. Find the maximum value of the following functions(show it's maximum):

a. f(x) = $(1+x)e^{\frac{-x}{2}}$ b. f(x) = x –x ln x

c. f(x) = xe^{-3x} – 2 d. f(x) = - $\sqrt{x^2+1}$

a. f(x) = $(1+x)e^{\frac{-x}{2}}$ of the form u.v

u = 1 + x $\Rightarrow$ u′ = 1 ; v = $e^{\frac{-x}{2}}$ $\Rightarrow$ v′ = $\frac{-1}{2}e^{\frac{-x}{2}}$

f′(x) = u′v + v′u = (1) $e^{\frac{-x}{2}}$ + $\frac{-1}{2}e^{\frac{-x}{2}}$ (1+x)

f′(x) = $e^{\frac{-x}{2}}\left(1-\frac{1+x}{2}\right)$ = $e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right)$ = 0 $\Rightarrow$ x = 1

To verify it is a maximum ,use second derivative test:

f′(x) = $e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right)$ $\Rightarrow$ f′′(x) = $\frac{-1}{2}e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right)+\frac{-1}{2}e^{\frac{-x}{2}}$

$\Rightarrow$ f′′(1) = 0 - $\frac{1}{2}e^{\frac{-1}{2}}$ < 0 $\Rightarrow$ x = 1 maximizes f(x) .

To find the maximum ,substitute x = 1 in f(x)

f(1) = (1+1) $e^{\frac{-1}{2}}$ =2$e^{\frac{-1}{2}}$ = $\frac{2}{\sqrt{e}}$

For the rest I will provide the final answers only :

b. f′(x) = 0 $\Rightarrow$ x = 1 ; f′′(1) = -1 < 0 ; Max = f(1) = 1

c. f′(x) = 0 $\Rightarrow$ x = $\frac{1}{3}$;f′′($\frac{1}{3}$) = -3e^{-1} < 0 ;Max = $\frac{1}{3}e^{-1}$ – 2

d. f′(x) = 0 $\Rightarrow$ x = 0 ; f′′(0) = -1 < 0 ; Max = f(0) = -1

6. Find the minimum value of the following functions(show it’s minimum) :

a. f(x) = 2x – ln x

b. f(x) = $x^2-\ln(\sqrt{2}\,x)$

c. f(x) = $e^x + e^{-x}$

d. f(x) = x^2 -2x + 5

a. f′(x) = 0 $\Rightarrow$ x = $\frac{1}{2}$; f′′($\frac{1}{2}$) = 4 > 0 ; Min = 1+ln2

b.f′(x) = 0 $\Rightarrow$ x = $\frac{\pm 1}{\sqrt{2}}$; f′′($\frac{1}{\sqrt{2}}$) = $\frac{4}{\sqrt{2}}$ > 0 ; Min = $\frac{1}{2}$

c. f′(x) = 0 $\Rightarrow$ x = 0 ; f′′(0) = 2 > 0 ; Min = f(0) = 2

d. f′(x) = 0 $\Rightarrow$ x = 1 ; f′′(1) = 2 > 0 ; Min = f(1) = 4