International Institute for Technology and Management

November 27th, 2005 **Unit: 05a – Mathematics 1**

GROUP(C)-VERSION C

This paper is not to be removed from the Examination Halls

Student Name :

Student Number :

Tuesday 27th November 7 : 00 pm – 9 : 00 pm

Candidates should answer **NINE** of the following **ELEVEN** questions: **SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

PLEASE TURN OVER

SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The supply equation for a good is

$$\mathbf{q = p^2 - 14p - 4}$$

and the demand equation is

$$\mathbf{q = -p^2 + 2p + 50}$$

where p is the price.

Sketch the supply and the demand functions for $\mathbf{p} \geq 0$

Determine the equilibrium price and quantity.

Supply $\mathbf{q = p^2 - 14p - 4}$

(1)It has U shape since it has positive p^2 term

(2)Intercepts: <u>p-intercepts</u> : q = 0 , $p^2 - 14p - 4 = 0$,

$$p = \frac{14 \pm \sqrt{212}}{2}$$

<u>q-intercpt</u>: p = 0 $\Rightarrow$ q = -4 ; (0,-4)

(3) The minimum : q' = 2p - 14 = 0 $\Rightarrow$ p = 7

$\Rightarrow$ q = 7^2 -14(7) -4 = -53; (7 , -53)

OR, p = $\frac{-b}{2a} = \frac{14}{2} = 7 \Rightarrow$ q =-53$\Rightarrow$ V(7 , -53)

Demand $\mathbf{q = -p^2 + 2p + 50}$

(1)It has $\bigcap$ shape since it has negative p^2 term

(2)Intercepts: <u>p-intercepts</u> : q = 0 , $-p^2 + 2p + 50 = 0$,

$$p = \frac{-2 \pm \sqrt{204}}{-2}$$

<u>q-intercpt</u>: p = 0 $\Rightarrow$ q = 50 ; (0,50)

(3) The minimum : q' = -2p +2 = 0 $\Rightarrow$ p = 1

$\Rightarrow$ q = 1^2 +2(1) +50= 53 ; (1 , 51)

OR, p = $\frac{-b}{2a} = \frac{-2}{-2} = 1 \Rightarrow$ q =53 $\Rightarrow$ V(1 , 51)

Equilibrium price and quantity

$\mathbf{p^2 - 14p - 4 = -p^2 + 2p + 50} \Rightarrow 2p^2 - 16p - 54 = 0$

4

$\Rightarrow p^2 - 8p - 27 = 0 \Rightarrow p = \frac{8 \pm \sqrt{172}}{2} = \frac{8 \pm \sqrt{4 \times 43}}{2}$

$$p = \frac{8 \pm 2\sqrt{43}}{2} = 4 \pm \sqrt{43}$$

p = $4 - \sqrt{43}$ which is economically not feasible, p = $4 + \sqrt{43}$

$\Rightarrow$ q = $(4 + \sqrt{43})^2 - 14(4 + \sqrt{43}) + 50 = 45 - 6\sqrt{43}$

Sketch of the graphs 5 marks each.

2. Find the value of x that maximises the function:

$$f(x) = (1+x)e^{-x/4}$$

Verify that it is a maximum.

$f'(x) = (1)\, e^{-x/4} - (1/4)(1+x)\, e^{-x/4} = e^{-x/4}(1 - ¼ - x/4)$

$f'(x) = ¼\, e^{-x/4}(3-x)$, $f'(x) = 0$, $3-x = 0 \Rightarrow x = 3$

$f''(x) = ¼\, [(-1)\, e^{-x/4} - (1/4)(3-x)\, e^{-x/4}] = ¼\, e^{-x/4}(-7+x/4)$

$f''(3) = ¼\, e^{-x/4}(-7+3/4) = -¼\, e^{-x/4} < 0$ 7

$\Rightarrow x = 3$ maximises f(x)

3. Determine the following integrals

$$\int \frac{dx}{x(2-\ln x)^3}$$, $u = 1-\ln x \Rightarrow du = -dx/x$

$$-\int \frac{du}{u^3} = -\int u^{-3} du = -\frac{u^{-2}}{-2} + C = \frac{1}{2u^2} + C = \frac{1}{2(1-\ln x)^2} + C$$ 5

$$\int \frac{\cos x}{\sqrt{1+\sin x}} dx$$, $u = 1+\sin x \Rightarrow du = \cos x\, dx$

$$\int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = 2\sqrt{u} + C = 2\sqrt{1+\sin x} + C$$ 5

4. A firm has average variable cost

$$q^2 + 2q + \frac{e^{q^2+1}}{q}$$

and fixed costs of 8 .Find the total cost function and the marginal cost function.

AVC = $q^2 + 2q + \frac{e^{q^2+1}}{q}$, AVC = $\frac{VC}{q}$, VC : the variable cost

VC = q x AVC = $q\left(q^2 + 2q + \frac{e^{q^2+1}}{q}\right) = q^3 + 2q^2 + e^{q^2+1}$

TC = VC + FC = $q^3 + 2q^2 + e^{q^2+1} + \mathbf{8}$ 6

MC = (TC)′ = $3q^2 + 4q + 2q\, e^{q^2+1}$

5. A firm has marginal cost qe^{q^2} and fixed costs are 5.
Find its total cost function.

TC = $\int MCdq = \int qe^{q^2} dq$, t = q^2 , dt = 2q dq

= ½ $\int e^t dt$ = ½ e^t + C = ½ e^{q^2} + C

FC = TC(0) $\Rightarrow$ ½ e^0 + C = 5 $\Rightarrow$ ½ + C = 5 , C = 9/2

TC = ½ e^{q^2} + 9/2

7

6. A monopolist's cost function is given by :

$q^2 - 1$

Where q is the quantity produced, the inverse demand function for the good is p = 32 − 7q

Determine expressions, in terms of q , for the revenue and The profit and determine the value of q that maximizes the profit. Find the maximum profit.

TR = pxq , but p = 32 − 7q

TR = (32 - 7q)xq = 32q − $7q^2$ 2

TC = $q^2 - 1$ 2

π = TR − TC = 32q − $7q^2$-q^2 +1 = -$8q^2$ +32q +1 2

π' = -16q +32 = 0 $\Rightarrow$ q = 2

π'' = -16 < 0 , $\Rightarrow$ q = 2 maximises the profit.

7. Determine the following integral

$$\int \frac{x+1}{x^2+2x+5}dx$$, u = x^2 + 2x + 5 , du = 2(x+1) dx

$$= ½ \int \frac{du}{u} = ½ \ln|u| + C = ½ \ln|x^2 + 2x + 5| + C$$

5

$\int \sin^2 x \cos^5 x \, dx$, u = sinx , du = cosx dx

$= \int \sin^2 x \cos^4 x \cos x \, dx$, but $\cos^2 x = 1 - \sin^2 x$

$= \int u^2(1 - \sin^2 x)^2 du = \int u^2(1-u^2)^2 du$

$= \int u^2(1 - 2u^2 + u^4)du = \int (u^2 - 2u^4 + u^6)du$

$$= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C = \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

5

SECTION B

Answer **TWO** questions from this section (20 marks each)

8. (a) A firm has average variable cost

$$2q^2 + 5q + \frac{\ln(q^3+2)}{q}$$

and fixed costs of 4 .Find the total cost function and the marginal cost function.

VC = q x AVC = $2q^3 + 5q^2 + \ln(q^3 + 2)$

TC = VC + FC = $2q^3 + 5q^2 + \ln(q^3 + 2) + 4$

MC = (TC)′ = $6q^2 + 10q + \dfrac{3q^2}{q^3+2}$ | 10 |

(b) Determine the following integrals

$\int x^2\sqrt{2x+1}\,dx$, $u^2 = 2x+1$, $2udu = 2dx$, $dx = udu$

= $\int x^2u^2du$, But $u^2 = 2x+1 \Rightarrow x = ½ (u^2 - 1)$

$\Rightarrow x^2 = ¼ (u^2 - 1)^2$

= ¼ $\int (u^2-1)^2u^2du$ = ¼ $\int (u^4-2u^2+1)u^2du$

= ¼ $\int (u^6-2u^4+u^2)du$ = ¼ ($u^7/7 - 2u^5/5 + u^3/3$) + C

= $\dfrac{(\sqrt{2x+1})^7}{28} - \dfrac{(\sqrt{2x+1})^5}{10} + \dfrac{(\sqrt{2x+1})^3}{12} + C$ | 5 |

$\int \dfrac{e^x}{(e^x+1)^2}dx$, $u = e^x + 1$, $du = e^x\,dx$

= $\int \dfrac{du}{u^2} = \int u^{-2}du = \dfrac{u^{-1}}{-1} + C = \dfrac{-1}{u} + C = \dfrac{-1}{e^x+1} + C$ | 5 |

9. (a) A firm's demand function is p = aq + b (a < 0 ; b > 0) fixed costs are c and variable costs are d per unit.

Show that the profit is maximized when q = $\dfrac{d-b}{2a}$

The profit π = TR − TC

TR = p x q and TC = VC + FC = ***dq + c***

Now p = aq + b $\Rightarrow$ TR = (aq + b) x q= $aq^2 + bq$

π = TR − TC = $aq^2 + bq - dq - c = aq^2 + (b-d)q - c$ | 10 |

$\pi' = 2aq + b - d = 0 \Rightarrow q = \dfrac{d-b}{2a}$

$\pi'' = 2a < 0$ (since a < 0) $\Rightarrow q = \dfrac{d-b}{2a}$ maximises the profit.

(b) Find the critical points of the function:

$$f(x) = x^4 - 8x^3 - 80x^2 + 15$$

and specify their nature.

$f`(x) = 4x^3 - 24x^2 - 160x = 0 \Rightarrow 4x(x^2 - 6x - 40) = 0$

$x = 0$, $x^2 - 6x - 40 = 0 \Rightarrow x = \dfrac{6 \pm \sqrt{196}}{2} = \dfrac{6 \pm 14}{2}$

$x = 10$, $x = -4$ **10**

$f''(x) = 12x^2 - 48x - 160$

at $x = 0$, $f''(0) = -160 < 0$, maximum : (0 , 15)

at $x = 10$, $f''(10) = 560 > 0$, minimum : (10 , 1215)

at $x = -4$, $f''(-4) = 224 > 0$, minimum : (-4 , -497)

10. (a) A firm's marginal revenue function is $MR = 11 - q$

The firm's marginal cost function is

$MC = q^2 - 3q + 3$

where q is either the quantity sold or produced.

Find the profit-maximizing level of output and verify that it is a maximum.

TC = $\int (q^2 - 3q + 3)dq$ = $q^3/3 - 3q^2/2 + 3q + C$ **2**

TR = $\int (11 - q)dq$ = $11q - q^2/2$ **2**

π = TR – TC = $11q - q^2/2 - q^3/3 + 3q^2/2 - 3q - C$ **2**

$\pi = -q^3/3 + q^2 + 8q - C \Rightarrow \pi' = -q^2 + 2q + 8 = 0$ **2**

q = - 2 which is economically not feasible, q = 4

$\pi'' = -2q + 2 = -2(4) + 2 = -6 < 0$, q = 4 maximises the profit **2**

<u>Another method</u>

MR = MC , $11 - q = q^2 - 3q + 3 \Rightarrow -q^2 + 2q + 8 = 0$

q = - 2 which economically not feasible, q = 4

(b) Determine the following integrals

$\int_1^2 \frac{(\ln x)^3}{x} dx$, u = lnx , du = dx/x

$= \int u^3 du = u^4/4 + C = (\ln x)^4/4 \Big|_1^2 = (\ln 2)^4/4 - (\ln 1)^4/4 = (\ln 2)^4/4 - 0$ **5**

$= 4(\ln 2)/4 = \ln 2$

$\int x^2 e^x dx$,integration by parts , $u = x^2$,$du = 2xdx$, $dv = e^x dx$ **5**

,$v = e^x$ $\int x^2 e^x dx = \int u dv = uv - \int v du = x^2 e^x - 2\int x e^x dx$

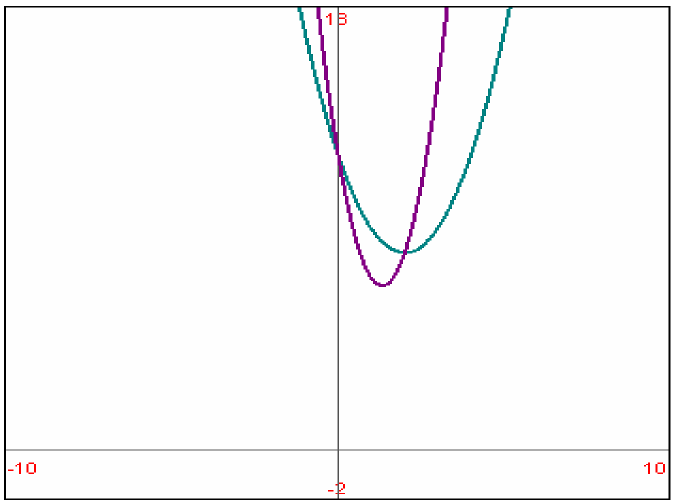
$= x^2 e^x - 2(xe^x - e^x) + C$ ($\int x e^x dx = xe^x - e^x + c$,By Parts)

11. A firm faces a total cost function TC = ½ q^3 – $4q^2$ + 6q

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

AC = $\frac{TC}{q}$ = ½ q^2 – 4q + 6 , MC = (TC)′ = 3/2 q^2 – 8q + 6 2

(ii) Sketch the average cost (AC) and the marginal cost(MC) on the same graph.



10

(iii) If price is \$ 15 , which level of output will a profit maximising firm choose?

TR = 15q

π = TR – TC = 15q – ½ q^3 +4q^2 -6q = – ½ q^3 +4q^2 +9q

π'= -3/2 q^2 + 8q +9 = 0 $\Rightarrow$ -3q^2 + 16q + 18 = 0

$$q = \frac{-16 \pm \sqrt{472}}{2} = \frac{6 \pm \sqrt{4 \times 118}}{2} = \frac{6 \pm 2\sqrt{118}}{2} = 3 \pm \sqrt{118}$$

$\Rightarrow$ q = $3 - \sqrt{118}$ which is economically not feasible , q = $3 + \sqrt{118}$ 8

π'' = -6q + 16 = -6($3 + \sqrt{118}$) + 16 < 0

$\Rightarrow$ q = $3 + \sqrt{118}$ maximises the profit

END OF PAPER