



November 29<sup>th</sup>, 2007

Unit: 05a – Mathematics 1

# GROUP(B)-VERSION B

This paper is not to be removed from the Examination Halls

## SOLUTION

Student Name :

Student Number :

Tuesday 27<sup>th</sup> November 13 : 30 pm – 15 : 30 pm

Candidates should answer **NINE** of the following **ELEVEN** questions: **SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

**PLEASE TURN OVER**

## SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The supply equation for a good is

$$q = p^2 - 2p + 3$$

and the demand equation is

$$q = -2p^2 + p + 9$$

where  $p$  is the price.

Sketch the supply and the demand functions for  $p \geq 0$

Determine the equilibrium price and quantity.

Supply  $q = p^2 - 2p + 3$

(1) It has U shape since it has positive  $p^2$  term

(2) Intercepts: p-intercepts :  $q = 0$

$$p^2 - 2p + 3 = 0, b^2 - 4ac = 4 - 4(1)(3) = -8 < 0$$

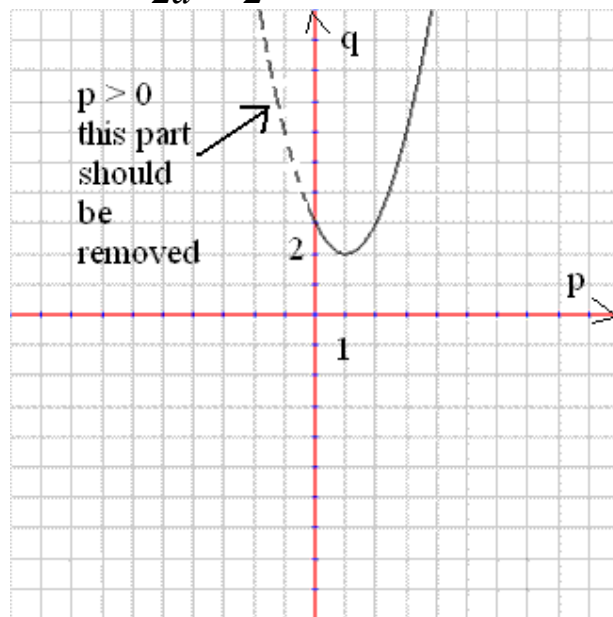
$\Rightarrow$  No intersection with the  $p$ -axis

q-intercept:  $p = 0 \Rightarrow q = 3$  ;  $(0, 3)$

(3) The minimum :  $q' = 2p - 2 = 0 \Rightarrow p = 1$

$\Rightarrow q = 2$  ;  $(1, 2)$

$$\text{OR } p = \frac{-b}{2a} = \frac{2}{2} = 1 \Rightarrow q = 2 \Rightarrow V(1, 2)$$



5

The demand :  $q = -2p^2 + p + 9$

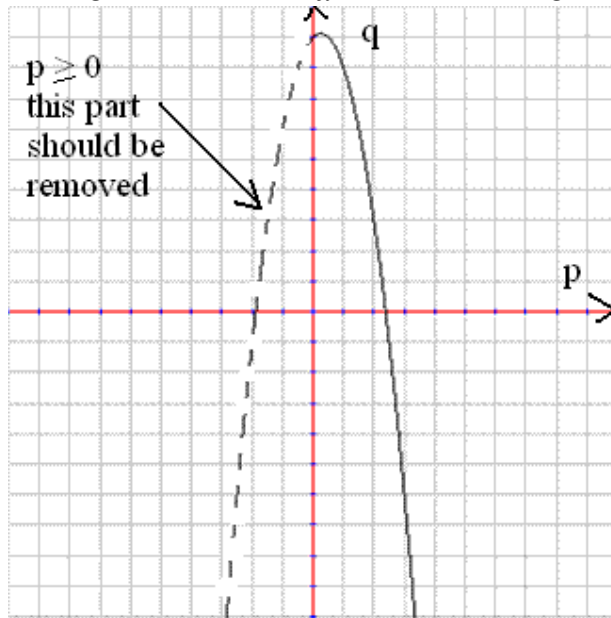
(1) It has  $\cap$  shape since it has negative  $p^2$  term .

(2) Intercepts: p-intercept ,  $q = 0 \Rightarrow -2p^2 + p + 9 = 0$

$$p = \frac{-1 \pm \sqrt{73}}{2} \Rightarrow \left( \frac{-1 - \sqrt{73}}{2}, 0 \right) \text{ and } \left( \frac{-1 + \sqrt{73}}{2}, 0 \right)$$

q-intercepts :  $p = 0 \Rightarrow q = 9$  ;  $(0, 9)$

(3) The maximum :  $q' = -4p + 1 = 0 \Rightarrow p = \frac{1}{4}$   
 $q = \frac{73}{8}$  OR  $p = \frac{-b}{2a} = \frac{1}{4} \Rightarrow q = \frac{73}{8} \Rightarrow V(\frac{1}{4}, \frac{73}{8})$



5

To determine the equilibrium price, we solve:  
 $p^2 - 2p + 3 = -2p^2 + p + 9 \Rightarrow 3p^2 - 3p - 6 = 0$   
 which is equivalent to  $p^2 - p - 2 = 0 \Rightarrow (p+1)(p-2) = 0$   
 Either  $p = -1$  or  $p = 2$  of which only  $p = 2$  is economically  
 Meaningful.  $p = 2 \Rightarrow q = 3$

4

2. Find the maximum value of the function, verify that it is a maximum:

$$f(x) = (x^2 - x + 1)e^{-x}$$

$$f'(x) = (2x-1)e^{-x} - (x^2 - x + 1)e^{-x} = (-x^2 + 3x - 2)e^{-x}$$

$$f'(x) = 0 \Rightarrow -x^2 + 3x - 2 = 0 \text{ (Since } e^x > 0 \forall x)$$

$$\Rightarrow \text{either } x = 1 \text{ or } x = 2$$

7

$$f''(x) = (-2x+3)e^{-x} - (-x^2 + 3x - 2)e^{-x} = (x^2 - 5x + 5)e^{-x}$$

$$f''(2) = -e^{-2} < 0 \Rightarrow x = 2 \text{ maximises } f(x).$$

3. Determine the following integrals  $\int \frac{(1 - \ln x)^2}{x} dx$

By substitution:  $t = 1 - \ln x \Rightarrow dt = \frac{-1}{x} dx$

5

$$\int \frac{(1 - \ln x)^2}{x} dx = -\int t^2 dt = -t^3/3 + C = -(1 - \ln x)^3 / 3 + C$$

$$\int \sin^3 x \cos^5 x dx$$

By substitution  $t = \cos x \Rightarrow dt = -\sin x dx \Rightarrow dx = -dt/\sin x$

$$\int \sin^3 x \cos^5 x dx = -\int \sin^3 x t^5 \frac{dt}{\sin x} = -\int \sin^2 x t^5 dt$$

But  $\sin^2 x = 1 - \cos^2 x = 1 - t^2$

$$= -\int (1 - t^2) t^5 dt = -\int (t^5 - t^7) dt = -t^6/6 + t^8/8 + C$$

$$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$$

5

4. The marginal cost for a company is

$$4q^3 + 6q + e^q - 1$$

and fixed costs of 60.

Find the total cost, the variable cost and the average cost functions.

$$TC = \int MC dq = \int (4q^3 + 6q + e^q - 1) dq$$

$$TC = q^4 + 3q^2 + e^q - q + C$$

$$FC = TC(0) = 60 \Rightarrow 0 + 0 + e^0 - 0 + C = 60 \Rightarrow 1 + C = 60$$

$$C = 59$$

$$TC = q^4 + 3q^2 + e^q - q + 59$$

3

$$TC = VC + FC \Rightarrow VC = TC - FC = q^4 + 3q^2 + e^q - q + 59 - 60$$

$$VC = q^4 + 3q^2 + e^q - q - 1$$

2

$$AC = \frac{TC}{q} = q^3 + 3q + \frac{e^q}{q} - 1 + \frac{59}{q}$$

2

5. (a) A firm's demand function is

$$p = aq + b \quad (a < 0 ; b > 0)$$

Fixed costs are  $c$  and variable costs are  $d$  per unit.

Show that the profit is maximized when  $q = \frac{d-b}{2a}$

The profit  $\pi = TR - TC$

$$TR = p \times q \quad \text{and} \quad TC = VC + FC = dq + c$$

$$\text{Now } p = aq + b \Rightarrow TR = (aq + b) \times q = aq^2 + bq$$

$$\pi = TR - TC = aq^2 + bq - dq - c = aq^2 + (b-d)q - c$$

$$\pi' = 2aq + b - d = 0 \Rightarrow q = \frac{d-b}{2a}$$

6

$$\pi'' = 2a < 0 \quad (\text{since } a < 0) \Rightarrow q = \frac{d-b}{2a} \text{ maximises the profit.}$$

6. A firm has average variable cost

$$q^2 + 7q + \frac{\ln(q^3 + 7)}{q}$$

and fixed costs of 7. Find the total cost function and the marginal cost function.

$$\begin{aligned} VC &= q \times AVC = q^3 + 7q^2 + \ln(q^3 + 7) \\ TC &= VC + FC = q^3 + 7q^2 + \ln(q^3 + 7) + 7 \end{aligned}$$

6

$$MC = (TC)' = 3q^2 + 14q + \frac{3q^2}{q^3 + 7}$$

7. Determine the following integrals

$$\int \frac{1}{x\sqrt{\ln x}(\ln x + 4\sqrt{\ln x} + 4)} dx, \quad t = \sqrt{\ln x} \Rightarrow dt = \frac{dx}{2x\sqrt{\ln x}}$$

$$2 \int \frac{1}{(t^2 + 4t + 4)} dt = 2 \int \frac{1}{(t + 2)^2} dt = 2 \int (t + 2)^{-2} dt$$

$$= \frac{-2}{t + 2} + C = \frac{-2}{\sqrt{\ln x} + 2} + C$$

5

$$\int \frac{\sqrt{1 + \tan x}}{\cos^2 x} dx, \quad t = 1 + \tan x \Rightarrow dt = dx / \cos^2 x$$

$$= \int \sqrt{t} dt = \int t^{1/2} dt = \frac{2t^{3/2}}{3} + C = \frac{2(1 + \tan x)^{3/2}}{3} + C$$

5

## SECTION B

Answer **TWO** questions from this section (20 marks each)

8. (a) A firm is a monopoly for the good it produces, It has a marginal cost function  $MC = 6q^2 + 8$  and fixed costs of 20. The demand equation for its good is given by  $p + 2q = 40$  where  $p$  is the price. Find expressions in terms of  $q$ , for the total revenue and profit. Determine the value of  $q$  that maximises the profit:

$$TR = pxq, \text{ but } p + 2q = 40 \Rightarrow p = 40 - 2q$$

$$TR = (40 - 2q)xq = 40q - 2q^2$$

2

$$TC = \int MC dq = \int (6q^2 + 8) dq = 2q^3 + 8q + C$$

$$FC = TC(0) = 20 \Rightarrow 0 + 0 + C = 20, C = 20$$

2

$$TC = 2q^3 + 8q + 20$$

$$\pi = TR - TC = 40q - 2q^2 - 2q^3 - 8q - 20 = -2q^3 - 2q^2 + 32q - 20$$

2

$$\pi' = -6q^2 - 4q + 32 = 0 \Rightarrow -3q^2 - 2q + 16 = 0$$

$$q = \frac{2 \pm \sqrt{4 - 4(-3)(16)}}{-6} = \frac{2 \pm \sqrt{4 + 192}}{-6} = \frac{2 \pm \sqrt{196}}{-6}$$

$$q = \frac{2 \pm 14}{-6} \Rightarrow q = 2, q = -16/6 = -8/3 \text{ which is economically}$$

2

not feasible, therefore  $q = 2$

$$\pi'' = -12q - 4, \pi''(2) = -12(2) - 4 = -28 < 0 \Rightarrow q = 2$$

2

maximises

The profit.

(b) Determine the following integrals

$$\int x^2 \sqrt{x^3 + 1} dx, t = x^3 + 1 \Rightarrow dt = 3x^2 dx \Rightarrow x^2 dx = dt/3$$

$$\int \sqrt{t} dt/3 = \frac{1}{3} \int t^{1/2} dt = \frac{1}{3} \times \frac{2t^{3/2}}{3} + C = \frac{2(x^3 + 1)^{3/2}}{9} + C$$

5

$$\int \frac{x}{\sqrt{x+1}} dx, t^2 = x + 1 \Rightarrow 2t dt = dx$$

$$\int \frac{x}{\sqrt{t^2}} \times 2t dt \text{ but, } t^2 = x + 1 \Rightarrow x = t^2 - 1$$

$$\int \frac{t^2 - 1}{t} \times 2t dt = 2 \int (t^2 - 1) dt = \frac{2t^3}{3} - 2t + C$$

$$\text{Now } t^2 = x + 1 \Rightarrow t = \sqrt{x + 1}$$

$$\frac{2t^3}{3} - 2t + C = \frac{2(\sqrt{x+1})^3}{3} - 2\sqrt{x+1} + C$$

5

9. (a) A firm is a monopoly its fixed costs are 20 it has average variable cost function **AVC = 10 + q** where q denotes its production level, the demand function of the good

$$\text{produced by firm is } q = 10 - \frac{p}{2}$$

where p is the price. Find expressions, in terms of q, for the revenue and the profit and determine the value of q that maximizes the profit. Calculate this maximum profit.

$$TR = pxq, \text{ but } q = 10 - p/2 \Rightarrow p = 20 - 2q$$

$$TR = (20 - 2q)xq = 20q - 2q^2$$

$$TC = VC + FC = q \times AVC + FC = 10q + q^2 + 20$$

$$\pi = TR - TC = 20q - 2q^2 - 10q - q^2 - 20 = -3q^2 + 10q - 20$$

$$\pi' = -6q + 10 = 0 \Rightarrow q = 10/6 = 5/3$$

$$\pi'' = -6 < 0, q = 5/3 \text{ maximises the profit.}$$

$$\text{Value of the maximum: } \pi(5/3) = -3(5/3)^2 + 10(5/3) - 20 = -85/3$$

2

2

2

2

- (b) The function  $f(x) = x^2 - \ln(\sqrt{2}x)$  is defined for  $x > 0$   
Determine the critical points of  $f$  and specify their nature.

2

$$f'(x) = 2x - \frac{1}{x} = \frac{2x^2 - 1}{x} = 0$$

2

$$\Rightarrow 2x^2 - 1 = 0 \Rightarrow x^2 = 1/2$$

2

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}} \quad x = -\frac{1}{\sqrt{2}} \text{ rejected since it doesn't}$$

belong to the domain of  $f$ .

2

$$x = \frac{1}{\sqrt{2}} \Rightarrow f(x) = 1/2 - \ln 1 = 1/2 \quad \text{the critical point } \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

2

$$f''(x) = 2 + \frac{1}{x^2} > 0 \text{ for every } x \Rightarrow x = \frac{1}{\sqrt{2}} \text{ minimises } f$$

- 10.(a) A firm's marginal revenue function is  $MR = 11 - q$

The firm's marginal cost function is

$$MC = q^2 - 3q + 3$$

where  $q$  is either the quantity sold or produced.

Find the profit-maximizing level of output and verify that it is a maximum.

$$TC = \int (q^2 - 3q + 3) dq = q^3/3 - 3q^2/2 + 3q + C$$

$$TR = \int (11 - q) dq = 11q - q^2/2$$

$$\pi = TR - TC = 11q - q^2/2 - q^3/3 + 3q^2/2 - 3q - C$$

$$\pi = -q^3/3 + q^2 + 8q - C \Rightarrow \pi' = -q^2 + 2q + 8 = 0$$

$$q = -2 \text{ which is economically not feasible, } q = 4$$

$$\pi'' = -2q + 2 = -2(4) + 2 = -6 < 0, \quad q = 4 \text{ maximises the profit}$$

Another method

$$MR = MC$$

$$11 - q = q^2 - 3q + 3 \Rightarrow -q^2 + 2q + 8 = 0$$

$$q = -2 \text{ which economically not feasible, } q = 4$$

- (b) Determine the following integrals  $\int \frac{x+1}{(x^2+2x+5)^2} dx$ ,

$$t = x^2 + 2x + 5 \Rightarrow dt = 2(x+1)dx \Rightarrow (x+1)dx = dt/2,$$

$$\int \frac{dt/2}{t^2} = \frac{1}{2} \int t^{-2} dt = \frac{-1}{2t} + C = \frac{-1}{2(x^2+2x+5)} + C$$

$$\int \frac{e^{-x} - e^x}{e^x + e^{-x}} dx, t = e^x + e^{-x} \Rightarrow dt = (e^x - e^{-x}) dx = -(e^{-x} - e^x) dx$$

$$\int \frac{-dt}{t} = -\ln|t| + C = -\ln(e^x + e^{-x}) + C$$

5

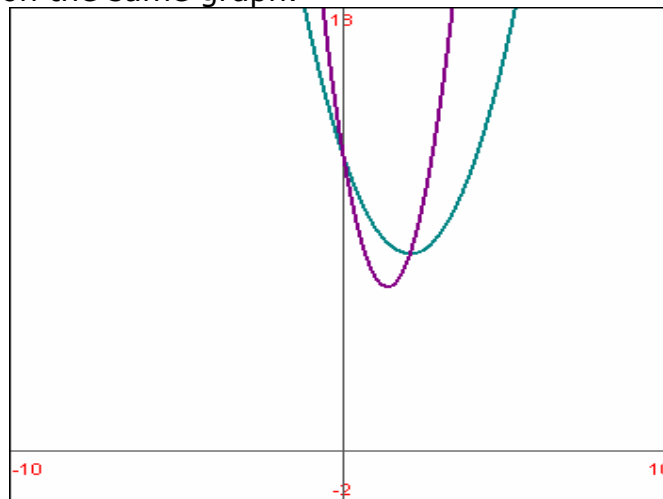
**11.** A firm faces a total cost function  $TC = q^3 - 4q^2 + 12q$

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

$$AC = \frac{TC}{q} = q^2 - 4q + 12, MC = (TC)' = 3q^2 - 8q + 12$$

2

(ii) Sketch the average cost (AC) and the marginal cost (MC) on the same graph.



10

(iii) If price is \$ 15, which level of output will a profit maximising firm choose?

8

$$TR = 15q$$

$$\pi = TR - TC = 15q - q^3 + 4q^2 - 12q = -q^3 + 4q^2 + 3q$$

$$\pi' = -3q^2 + 8q + 3 = 0 \Rightarrow q = -1/3 \text{ which is economically not feasible, } q = 3$$

$$\pi'' = -6q + 8 = -6(3) + 8 = -10 < 0 \Rightarrow q = 3 \text{ maximises the profit}$$

**END OF ANSWERS**