**International Institute for Technology and Management**

**November 2008** **Unit: 05a – Mathematics 1**

# GROUP(B)-VERSION B

This paper is not to be removed from the Examination Halls

**Student Name :**

**Student Number :**

TIME ALLOWED: 2 hours

Candidates should answer **NINE** of the following **ELEVEN** questions: **SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

**PLEASE TURN OVER**

## SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

**1.** The supply equation for a good is

$$q = p^2 - 4$$

and the demand equation is

$$q = p + 2$$

where p is the price.

Sketch the supply and the demand functions for $p \geq 0$

Determine the equilibrium price and quantity.

Supply $q = p^2 - 4$

(1) It has U shape since it has positive $p^2$ term

(2) Intercepts: p-intercepts : q = 0

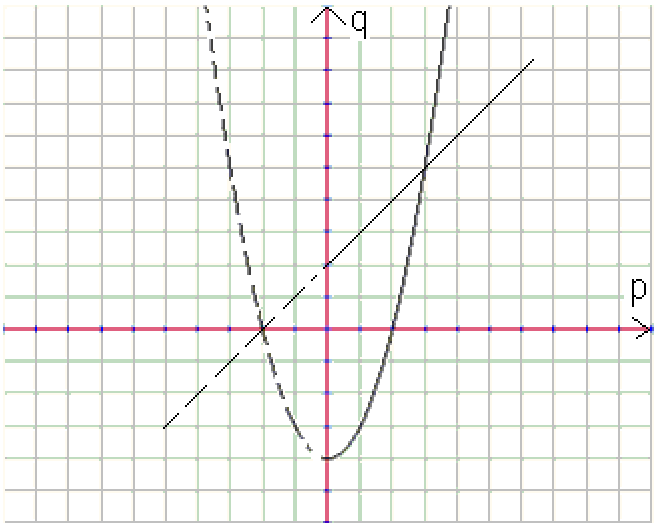
$p^2 - 4 = 0$ , $\Rightarrow$ p=-2 or p=2 ; (-2,0) and (2,0)

q-intercpt: p = 0 $\Rightarrow$ q = -4 ; (0,-4)

(3) The minimum : q' = 2p = 0 $\Rightarrow$ p = 0

$\Rightarrow$ q = -4 ; (0 , -4)

OR $p = \dfrac{-b}{2a} = \dfrac{0}{2} = 0 \Rightarrow$ q =-4 $\Rightarrow$ V(0,-4)

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The demand : $q = p + 2$

Intercepts :

p-intercept : q = 0 $\Rightarrow$ p = - 2 ; (-2,0)

q-intercept : p = 0 $\Rightarrow$ q = 2 ;(0 , 2)

2

To determine the equilibrium price, we solve:

$p^2 - 4 = p + 2 \Rightarrow p^2 - p - 6 = 0 \Rightarrow (p+2)(p-3) = 0$

Either p = - 2 or p=3 of which only p=3 is economically Meaningful. p=3 $\Rightarrow$ q = 5

2

**2.** Find the critical points of the following function:

$$f(x) = x^3e^{-x}$$

and specify their nature.

$f'(x) = 3x^2e^{-x} - x^3e^{-x} = 0 \Rightarrow (3x^2 - x^3)e^{-x} = 0$

but $e^{-x} \neq 0 \Rightarrow 3x^2 - x^3 = 0 \Rightarrow x^2(3 - x) = 0$

$x = 0 \Rightarrow y = 0$ , first point (0.0) [8]

$x=3 \Rightarrow y = 27e^{-3}$ , second point $(3,27e^{-3})$

$f'' = (6x - 3x^2)e^{-x} - (3x^2 - x^3)e^{-x} = (6x - 6x^2 + x^3)e^{-x}$

at x = 0 , f''(0) = 0 , **the test fails** and we need to study the sign of $f' = x^2(3-x)e^{-x}$ , since $x^2e^{-x} > 0$ , you may ignore it and study the sign of 3 – x

| | **0** | | **3** | |
|---|---|---|---|---|
| + + + | \| | + + | \| | - - - |
| | | ↗ (increasing) | | ↘ (decreasing) |

Notice at x=0 , f' does not change the sign, so it is a point of **inflection**.

At x= 3 , it maximizes f

Another way is $f''(3) = -9e^{-3} < 0 \Rightarrow x = 3$ maximizes f

**3.** Determine the following integrals

$\int x\sqrt{1-x}\,dx$ Let u = 1 – x , du = - dx

$= -\int x\sqrt{u}\,du$ now u = 1 – x , x = 1 - u

$= -\int(1-u)u^{1/2}\,du = -\int(u^{1/2} - u^{3/2})\,du = \int(u^{3/2} - u^{1/2})\,du$

$= \dfrac{2u^{5/2}}{5} - \dfrac{2u^{3/2}}{3} + C = \dfrac{2(1-x)^{5/2}}{5} - \dfrac{2(1-x)^{3/2}}{3} + C$ [5]

$\int \dfrac{dx}{x(2-\ln x)^3}$ Let u = 2 - lnx , du = -dx/x , $-\int \dfrac{du}{u^3}$

$= -\int u^{-3}du = -\dfrac{u^{-2}}{-2} + C = \dfrac{1}{2u^2} + C = \dfrac{1}{2(2-\ln x)^2} + C$ [5]

**4.** The total cost function of a firm is

$$\mathbf{5000 + 15q +} q^2\sqrt{1+2q}$$

Find the average cost and the marginal cost functions.

$AC = TC/q = 5000/q + 15 + q\sqrt{1+2q}$ [2]

$MC = \dfrac{d}{dq}TC = 0 + 15 + 2q(\sqrt{1+2q}) + q^2\left(\dfrac{2}{2\sqrt{1+2q}}\right)$

$MC = 15 + 2q(\sqrt{1+2q}) + q^2\left(\dfrac{1}{\sqrt{1+2q}}\right)$ [4]

**5.** Find the positive number a which is such that

$$\int_1^a \left(1+\frac{2}{x^2}\right)dx = 2$$

$$\int_1^a \left(1+2x^{-2}\right)dx = 2 \Rightarrow \left(x+\frac{2x^{-2+1}}{-2+1}\right)_1^a = 2 \Rightarrow \left(x-\frac{2}{x}\right)_1^a = 2$$

$$\Rightarrow \left(a-\frac{2}{a}\right)-\left(1-\frac{2}{1}\right) = 2 \Rightarrow a-\frac{2}{a} + 1 = 2 \Rightarrow a^2 - a - 2 = 0$$

$\Rightarrow$ a = 2 or a = -1 which is rejected since a > 0 $\Rightarrow$ a = 2

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**6.** The demand function for a company is

$q = 1000e^{-0.2\,p}$

the fixed costs are 100 and the variable costs are 2 per unit. Find the price needed to maximize the profit.

VC = 2q , FC = 100 $\Rightarrow$ TC = VC + FC = 2q + 100

TR=pq but $q=1000e^{-0.2\,p} \Rightarrow \frac{q}{1000} = e^{-0.2\,p} \Rightarrow 0.001q=e^{-0.2\,p}$

$\text{Ln}(0.001q) = \ln e^{-0.2p} \Rightarrow -0.2p = \text{Ln}(0.001q) \Rightarrow p = \frac{-1}{0.2}\text{Ln}(0.001q)$

$\Rightarrow$ p = -5 Ln(0.001q) , Now TR = pq = (-5 Ln(0.001q))**q**

$\pi$ = TR – TC = -5q **Ln(0.001q)** – 2q – 100

The first part : -5q **Ln(0.001q)** is of the uxv

u = -5q , u' = - 5

v = Ln(0.001q) , $v' = \frac{1}{0.001q} \times 0.001 = \frac{1}{q}$

its derivative is u'v + v'u

10

$\pi'$ = -5Ln(0.001q) **-5q(1/q)** – 2 = 0

$\Rightarrow$ -5Ln(0.001q) – 7 = 0

$\Rightarrow$ Ln(0.001q) = -7/5 $\Rightarrow$ $0.001q = e^{-7/5}$ $\Rightarrow$ **$q = 1000e^{-7/5}$**

**Now** $p = -5\,\text{Ln}(0.001q) = -5\ln(0.001 \times 1000e^{-7/5}) = -5\ln e^{-7/5}$

**p = -5(-7/5) = 7 .**

**Remember $\ln e^{-7/5} = (-7/5)\ln e = -7/5$ since $\ln e = 1$**

**7.** Determine the following integrals

$$\int \frac{x\ln(1+x^2)}{1+x^2}dx$$ Let u =ln(1+$x^2$) , du = $\frac{2x}{1+x^2}dx$

$$\Rightarrow \frac{x}{1+x^2}dx = \frac{1}{2}du$$ and the integral becomes

$$\frac{1}{2}\int udu = \frac{1}{2}\frac{u^2}{2} + C = \frac{u^2}{4} + C = \frac{\ln^2(1+x^2)}{4} + C$$ 5

$$\int \frac{\sin xdx}{\cos^4 x}$$ Let u = cosx , du = -sinx dx , sinxdx = -du

= $\int \frac{-du}{u^4}$ = $-\int u^{-4}du = -\frac{u^{-3}}{-3} + C = \frac{1}{3u^3} + C = \frac{1}{3\cos^3 x} + C$ 5

## SECTION B

Answer **TWO** questions from this section (20 marks each)

**8.** **(a)** A firm is a monopoly for the good it produces, It has a marginal cost function **MC = q - 5** and fixed costs of **300**.The demand equation for its good is given by **q = 850 – 10p** where p is the price. Find expressions in terms of q, for the total revenue and profit. Determine the value of q that maximises the profit. Calculate this maximum profit.

(a) TC = $\int MCdq$ = $\int (q-5)dq$ = ½ $q^2$ - 5q + C

FC = TC(0) = 300 $\Rightarrow$ 0 + 0 + C = 300 , C = 300

TC = ½ $q^2$ - 5q + 300

TR = pq , but **q = 850 – 10p** $\Rightarrow$ p = 85 – (1/10)q

TR = [85 – (1/10)q]q = 85q – (1/10)$q^2$

$\pi$ = TR – TC = 85q –(1/10)$q^2$ - ½ $q^2$ + 5q - 300

$\pi$ = -(6/10)$q^2$ +90q – 300

$\pi'$ = -(6/5)q +90 = 0 10

q = $\frac{-90}{-6/5}$ = 75 and $\pi''$ = -6/5 < 0

$\Rightarrow$ q = 75 maximizes the profit and the maximum profit is

$\pi$(75) = -(6/10)$(75)^2$ +90(75) – 300

= (75)[ (- 6/10)(75) + 90] – 300 = (75)(-45+90)-300

= (75)(45) – 300 = 3375 – 300 =3075

**(b)** Determine the following integrals

$\int \frac{x+3}{(x^2+6x+5)^2}dx$ Let u = $x^2$ +6x+5 , du = (2x+6)dx

du = 2(x+3)dx $\Rightarrow$ (x+3)dx = du/2

= $\int \frac{du/2}{u^2} = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\times\frac{u^{-1}}{-1} + C = \frac{-1}{2u} + C$

= $\frac{-1}{2(x^2+6x+5)} + C$ 5

$\int \frac{dx}{e^x(2+e^{-x})}$ Let u = 2+ $e^{-x}$ $\Rightarrow$ du = - $e^{-x}$ dx

= $\int \frac{e^{-x}dx}{(2+e^{-x})}$ = $\int \frac{-du}{u}$ = - ln u + C = -ln (2+$e^{-x}$) + C 5

**9.** **(a)** A monopolist's cost function is given by : $q^2 - 1$
Where q is the quantity produced, the inverse demand function for the good is p = 32 – 7q
Determine expressions, in terms of q , for the revenue and The profit and determine the value of q that maximizes the profit. Find the maximum profit.

TC = $q^2 - 1$
TR = pq , but p = 32 – 7q
TR = (32 – 7q)q = 32q – $7q^2$
$\pi$ = TR – TC = 32q – $7q^2$ – $q^2$ +1 = -$8q^2$ +32q +1
$\pi'$ = -16q +32 = 0 $\Rightarrow$ q = 2 10
$\pi''$ = -16 < 0
$\Rightarrow$ q = 2 maximizes the profit and the maximum profit is
$\pi$(2) = -32 + 64 +1 = 33

**(b)** Find and classify the stationary points of the function
***f*(x) = 3x⁴ – 4x³ -12x² + 9** ($f(x) = 3x^4 - 4x^3 - 12x^2 + 9$)

f'(x) = $12x^3 - 12x^2$ - 24x = 0 $\Rightarrow$ 12x($x^2$ – x – 2 ) = 0
$\Rightarrow$ 12x = 0 $\Rightarrow$ x = 0 $\Rightarrow$ y = 9 $\Rightarrow$ 1st point **(0 , 9)**
or $x^2$ – x – 2 = 0 $\Rightarrow$ x = -1 or x = 2
x = -1 $\Rightarrow$ y = $3(-1)^4 -4(-1)^3 -12(-1)^2$ + 9 = 4 $\Rightarrow$ 2nd point (-1 , 4)
x = 2 $\Rightarrow$ y = $3(2)^4 -4(2)^3 -12(2)^2$+9=-23 $\Rightarrow$ 3rd point ( 2 , -23) 10
Nature: f''(x) = $36x^2$ – 24x - 24
at x = 0 $\Rightarrow$ f''(0) = -24 < 0 $\Rightarrow$ (0 , 9) maximizes f(x)
at x = -1 $\Rightarrow$ f''(-1) = 36 > 0 $\Rightarrow$ (-1 , 4) minimizes f(x)
at x = 2 $\Rightarrow$ f''(2) = 72 > 0 $\Rightarrow$ ( 2 , -23) minimizes f(x)

**10.** **(a)** A firm's marginal revenue function is given by:

MR= **3 / 2** . The firm's total cost function is given by:

TC = 6 + q + 4q$^{2}$ + 6q$^{3}$

where q is either the quantity sold or produced.

Find the quantity that maximises the profit and verify that it is a maximum.

MC = $\frac{d}{dq}(TC)$ = $1+8q+18q^2$

Profit is maximum : MC = MR

$1+8q+18q^2 = \frac{3}{2} \Rightarrow 36q^2 + 16q - 1 = 0$

$$q = \frac{-16 \pm \sqrt{400}}{72} = \frac{-16 \pm 20}{72}$$

| 10 |
|---|

Either q = $\frac{-1}{2}$ or q = $\frac{4}{72} = \frac{1}{18}$ Of which only q = $\frac{1}{18}$ is economically meaningful.

**Another Method :**

MR = $\frac{3}{2} \Rightarrow$ TR = $\int MR dq = \int \frac{3}{2} dq = \frac{3q}{2} + C$

C = TR(0) = 0 $\Rightarrow$ MR = $\frac{3q}{2}$

TC = 6 + q + 4q$^{2}$ + 6q$^{3}$

Profit : $\pi$ = TR – TC = $\frac{3q}{2}$-(6 + q + 4q$^{2}$ + 6q$^{3}$)

$\pi = -6q^3 - 4q^2 + \frac{q}{2} - 6$ ; $\frac{d\pi}{dq} = 0 \Rightarrow -18q^2 - 8q + \frac{1}{2} = 0$

$\Rightarrow 36q^2 + 16q - 1 = 0$ , same result as above **q =** $\frac{1}{18}$

**(b)** **Show that :** $\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

$$x + \frac{x}{x^2-1} = \frac{x(x^2-1)+x}{x^2-1} = \frac{x^3}{x^2-1}$$

**Then find :** $\int \frac{x^3}{x^2-1}$ **dx =** $\int (x + \frac{x}{x^2-1})dx$

| 10 |
|---|

**= ½ x$^{2}$ +** $\int \frac{x}{x^2-1}dx$ **Let u =x$^{2}$ – 1 , du = 2x dx , xdx= ½ du**

**= ½ x$^{2}$ +** $\frac{1}{2}\int \frac{du}{u}$ **= ½ x$^{2}$+ ½ ln**$|u|$ **+ C = ½ x$^{2}$+ ½ ln**$|x^2-1|$ **+ C**

**11.** A firm faces a total cost function $TC = 20 + 5q + 5q^2$

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

$$AC = \frac{TC}{q} = \frac{20}{q} + 5 + 5q \quad , \quad MC = TC' = 5 + 10q$$

4

(ii) Find the quantity that minimises the Average cost and the value of this minimum. Show indeed it is a minimum.

$$AC' = \frac{-20}{q^2} + 5 = \frac{5q^2 - 20}{q^2}$$

6

$AC' = 0 \Rightarrow 5q^2 - 20 = 0 \Rightarrow q^2 = 4 \Rightarrow q = -2$ (rejected) or $q = 2$

$AC'' = 40/q^3$ and $AC''(2) = 40/8 = 5 > 0 \Rightarrow q = 2$ minimizes AC and the value of the minimum is

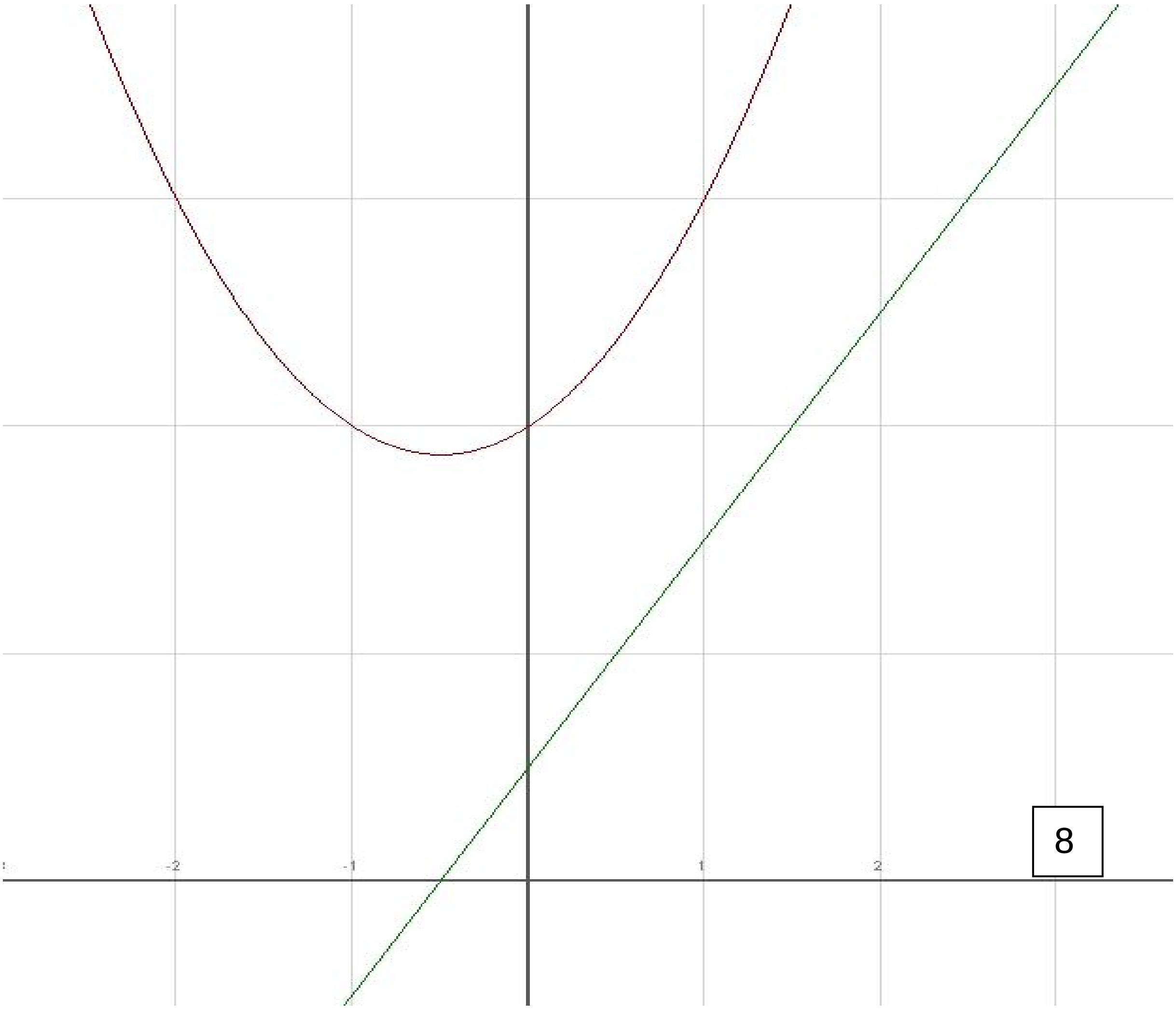
$AC(2) = 10 + 5 + 10 = 25$

(iii) Verify that when q = 2 , the marginal cost MC equals the Average cost.

2

$AC(2) = 25$ , $MC(2) = 5 + 20 = 25 \therefore AC = MC$ at $q = 2$

(iv) Sketch the graphs of the total cost TC and the marginal cost MC functions on the same system of axes.

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