



November 2008

Unit: 05a – Mathematics 1

GROUP(B)-VERSION B

This paper is not to be removed from the Examination Halls

Student Name :

Student Number :

TIME ALLOWED: 2 hours

Candidates should answer **NINE** of the following **ELEVEN** questions: **SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

PLEASE TURN OVER

SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The supply equation for a good is

$$q = p^2 - 4$$

and the demand equation is

$$q = p + 2$$

where p is the price.

Sketch the supply and the demand functions for $p \geq 0$

Determine the equilibrium price and quantity.

Supply $q = p^2 - 4$

(1) It has U shape since it has positive p^2 term

(2) Intercepts: p-intercepts : $q = 0$

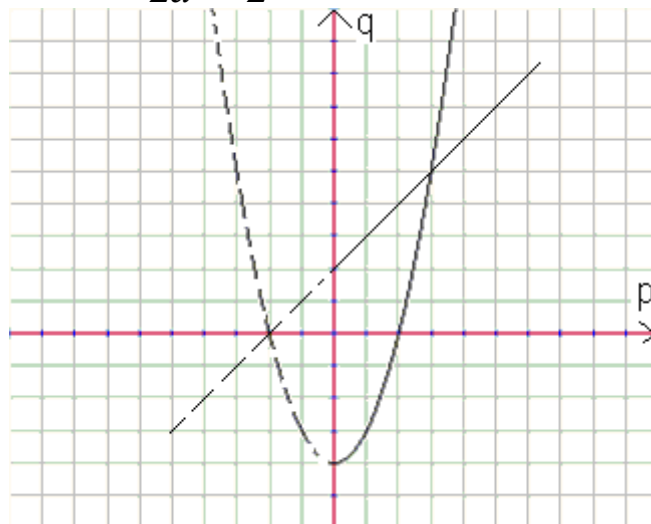
$$p^2 - 4 = 0, \Rightarrow p = -2 \text{ or } p = 2; (-2, 0) \text{ and } (2, 0)$$

q-intercept: $p = 0 \Rightarrow q = -4; (0, -4)$

(3) The minimum : $q' = 2p = 0 \Rightarrow p = 0$

$$\Rightarrow q = -4; (0, -4)$$

$$\text{OR } p = \frac{-b}{2a} = \frac{0}{2} = 0 \Rightarrow q = -4 \Rightarrow V(0, -4)$$



6

The demand : $q = p + 2$

Intercepts :

p-intercept : $q = 0 \Rightarrow p = -2; (-2, 0)$

q-intercept : $p = 0 \Rightarrow q = 2; (0, 2)$

2

To determine the equilibrium price, we solve:

$$p^2 - 4 = p + 2 \Rightarrow p^2 - p - 6 = 0 \Rightarrow (p+2)(p-3) = 0$$

Either $p = -2$ or $p = 3$ of which only $p = 3$ is economically

Meaningful. $p = 3 \Rightarrow q = 5$

2

5. Find the positive number a which is such that

$$\int_1^a \left(1 + \frac{2}{x^2}\right) dx = 2$$

$$\int_1^a (1 + 2x^{-2}) dx = 2 \Rightarrow \left(x + \frac{2x^{-2+1}}{-2+1}\right)_1^a = 2 \Rightarrow \left(x - \frac{2}{x}\right)_1^a = 2$$

$$\Rightarrow \left(a - \frac{2}{a}\right) - \left(1 - \frac{2}{1}\right) = 2 \Rightarrow a - \frac{2}{a} + 1 = 2 \Rightarrow a^2 - a - 2 = 0$$

$$\Rightarrow a = 2 \text{ or } a = -1 \text{ which is rejected since } a > 0 \Rightarrow a = 2$$

6

6. The demand function for a company is

$$q = 1000e^{-0.2p}$$

the fixed costs are 100 and the variable costs are 2 per unit. Find the price needed to maximize the profit.

$$VC = 2q, FC = 100 \Rightarrow TC = VC + FC = 2q + 100$$

$$TR = pq \text{ but } q = 1000e^{-0.2p} \Rightarrow \frac{q}{1000} = e^{-0.2p} \Rightarrow 0.001q = e^{-0.2p}$$

$$\ln(0.001q) = \ln e^{-0.2p} \Rightarrow -0.2p = \ln(0.001q) \Rightarrow p = \frac{-1}{0.2} \ln(0.001q)$$

$$\Rightarrow p = -5 \ln(0.001q), \text{ Now } TR = pq = (-5 \ln(0.001q))q$$

$$\pi = TR - TC = -5q \ln(0.001q) - 2q - 100$$

The first part : $-5q \ln(0.001q)$ is of the uvx

$$u = -5q, u' = -5$$

$$v = \ln(0.001q), v' = \frac{1}{0.001q} \times 0.001 = \frac{1}{q}$$

its derivative is $u'v + v'u$

10

$$\pi' = -5 \ln(0.001q) - 5q(1/q) - 2 = 0$$

$$\Rightarrow -5 \ln(0.001q) - 7 = 0$$

$$\Rightarrow \ln(0.001q) = -7/5 \Rightarrow 0.001q = e^{-7/5} \Rightarrow q = 1000e^{-7/5}$$

$$\text{Now } p = -5 \ln(0.001q) = -5 \ln(0.001 \times 1000e^{-7/5}) = -5 \ln e^{-7/5}$$

$$p = -5(-7/5) = 7.$$

$$\text{Remember } \ln e^{-7/5} = (-7/5) \ln e = -7/5 \text{ since } \ln e = 1$$

7. Determine the following integrals

$$\int \frac{x \ln(1+x^2)}{1+x^2} dx \quad \text{Let } u = \ln(1+x^2) \quad , \quad du = \frac{2x}{1+x^2} dx$$

$$\Rightarrow \frac{x}{1+x^2} dx = \frac{1}{2} du \quad \text{and the integral becomes}$$

$$\frac{1}{2} \int u du = \frac{1}{2} \frac{u^2}{2} + C = \frac{u^2}{4} + C = \frac{\ln^2(1+x^2)}{4} + C \quad \boxed{5}$$

$$\int \frac{\sin x dx}{\cos^4 x} \quad \text{Let } u = \cos x \quad , \quad du = -\sin x dx \quad , \quad \sin x dx = -du$$

$$= \int \frac{-du}{u^4} = - \int u^{-4} du = - \frac{u^{-3}}{-3} + C = \frac{1}{3u^3} + C = \frac{1}{3\cos^3 x} + C \quad \boxed{5}$$

SECTION B

Answer **TWO** questions from this section (20 marks each)

8. (a) A firm is a monopoly for the good it produces, It has a marginal cost function **MC = q - 5** and fixed costs of **300**. The demand equation for its good is given by **q = 850 - 10p** where p is the price. Find expressions in terms of q, for the total revenue and profit. Determine the value of q that maximises the profit. Calculate this maximum profit.

$$(a) \quad TC = \int MC dq = \int (q - 5) dq = \frac{1}{2} q^2 - 5q + C$$

$$FC = TC(0) = 300 \Rightarrow 0 + 0 + C = 300 \quad , \quad C = 300$$

$$TC = \frac{1}{2} q^2 - 5q + 300$$

$$TR = pq \quad , \quad \text{but } q = 850 - 10p \Rightarrow p = 85 - (1/10)q$$

$$TR = [85 - (1/10)q]q = 85q - (1/10)q^2$$

$$\pi = TR - TC = 85q - (1/10)q^2 - \frac{1}{2} q^2 + 5q - 300$$

$$\pi = -(6/10)q^2 + 90q - 300$$

$$\pi' = -(6/5)q + 90 = 0$$

10

$$q = \frac{-90}{-6/5} = 75 \quad \text{and} \quad \pi'' = -6/5 < 0$$

$\Rightarrow q = 75$ maximizes the profit and the maximum profit is

$$\pi(75) = -(6/10)(75)^2 + 90(75) - 300$$

$$= (75)[(-6/10)(75) + 90] - 300 = (75)(-45 + 90) - 300$$

$$= (75)(45) - 300 = 3375 - 300 = 3075$$

(b) Determine the following integrals

$$\int \frac{x+3}{(x^2+6x+5)^2} dx \quad \text{Let } u = x^2+6x+5, \quad du = (2x+6)dx$$

$$du = 2(x+3)dx \Rightarrow (x+3)dx = du/2$$

$$= \int \frac{du/2}{u^2} = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \times \frac{u^{-1}}{-1} + C = \frac{-1}{2u} + C$$

$$= \frac{-1}{2(x^2+6x+5)} + C \quad \boxed{5}$$

$$\int \frac{dx}{e^x(2+e^{-x})} \quad \text{Let } u = 2+e^{-x} \Rightarrow du = -e^{-x} dx$$

$$= \int \frac{e^{-x} dx}{(2+e^{-x})} = \int \frac{-du}{u} = -\ln u + C = -\ln(2+e^{-x}) + C \quad \boxed{5}$$

9. (a) A monopolist's cost function is given by : $q^2 - 1$
 Where q is the quantity produced, the inverse demand function for the good is $p = 32 - 7q$
 Determine expressions, in terms of q , for the revenue and The profit and determine the value of q that maximizes the profit. Find the maximum profit.

$$TC = q^2 - 1$$

$$TR = pq, \quad \text{but } p = 32 - 7q$$

$$TR = (32 - 7q)q = 32q - 7q^2$$

$$\pi = TR - TC = 32q - 7q^2 - q^2 + 1 = -8q^2 + 32q + 1$$

$$\pi' = -16q + 32 = 0 \Rightarrow q = 2 \quad \boxed{10}$$

$$\pi'' = -16 < 0$$

$$\Rightarrow q = 2 \text{ maximizes the profit and the maximum profit is}$$

$$\pi(2) = -32 + 64 + 1 = 33$$

(b) Find and classify the stationary points of the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 9$$

$$f'(x) = 12x^3 - 12x^2 - 24x = 0 \Rightarrow 12x(x^2 - x - 2) = 0$$

$$\Rightarrow 12x = 0 \Rightarrow x = 0 \Rightarrow y = 9 \Rightarrow 1^{\text{st}} \text{ point } (0, 9)$$

$$\text{or } x^2 - x - 2 = 0 \Rightarrow x = -1 \text{ or } x = 2$$

$$x = -1 \Rightarrow y = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 9 = 4 \Rightarrow 2^{\text{nd}} \text{ point } (-1, 4)$$

$$x = 2 \Rightarrow y = 3(2)^4 - 4(2)^3 - 12(2)^2 + 9 = -23 \Rightarrow 3^{\text{rd}} \text{ point } (2, -23) \quad \boxed{10}$$

$$\text{Nature: } f''(x) = 36x^2 - 24x - 24$$

$$\text{at } x = 0 \Rightarrow f''(0) = -24 < 0 \Rightarrow (0, 9) \text{ maximizes } f(x)$$

$$\text{at } x = -1 \Rightarrow f''(-1) = 36 > 0 \Rightarrow (-1, 4) \text{ minimizes } f(x)$$

$$\text{at } x = 2 \Rightarrow f''(2) = 72 > 0 \Rightarrow (2, -23) \text{ minimizes } f(x)$$

10. (a) A firm's marginal revenue function is given by:

MR = $\frac{3}{2}$. The firm's total cost function is given by:

$$TC = 6 + q + 4q^2 + 6q^3$$

where q is either the quantity sold or produced.

Find the quantity that maximises the profit and verify that it is a maximum.

$$MC = \frac{d}{dq}(TC) = 1 + 8q + 18q^2$$

Profit is maximum : MC = MR

$$1 + 8q + 18q^2 = \frac{3}{2} \Rightarrow 36q^2 + 16q - 1 = 0$$

$$q = \frac{-16 \pm \sqrt{400}}{72} = \frac{-16 \pm 20}{72}$$

10

Either $q = \frac{-1}{2}$ or $q = \frac{4}{72} = \frac{1}{18}$ Of which only $q = \frac{1}{18}$

is economically meaningful.

Another Method :

$$MR = \frac{3}{2} \Rightarrow TR = \int MR dq = \int \frac{3}{2} dq = \frac{3q}{2} + C$$

$$C = TR(0) = 0 \Rightarrow MR = \frac{3q}{2}$$

$$TC = 6 + q + 4q^2 + 6q^3$$

$$\text{Profit : } \pi = TR - TC = \frac{3q}{2} - (6 + q + 4q^2 + 6q^3)$$

$$\pi = -6q^3 - 4q^2 + \frac{q}{2} - 6 ; \frac{d\pi}{dq} = 0 \Rightarrow -18q^2 - 8q + \frac{1}{2} = 0$$

$$\Rightarrow 36q^2 + 16q - 1 = 0, \text{ same result as above } q = \frac{1}{18}$$

(b) Show that : $\frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1}$

$$x + \frac{x}{x^2 - 1} = \frac{x(x^2 - 1) + x}{x^2 - 1} = \frac{x^3}{x^2 - 1}$$

$$\text{Then find : } \int \frac{x^3}{x^2 - 1} dx = \int \left(x + \frac{x}{x^2 - 1} \right) dx$$

10

$$= \frac{1}{2} x^2 + \int \frac{x}{x^2 - 1} dx \quad \text{Let } u = x^2 - 1, du = 2x dx, x dx = \frac{1}{2} du$$

$$= \frac{1}{2} x^2 + \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} x^2 + \frac{1}{2} \ln|u| + C = \frac{1}{2} x^2 + \frac{1}{2} \ln|x^2 - 1| + C$$

11. A firm faces a total cost function $TC = 20 + 5q + 5q^2$

- (i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

$$AC = \frac{TC}{q} = \frac{20}{q} + 5 + 5q, \quad MC = TC' = 5 + 10q$$

4

- (ii) Find the quantity that minimises the Average cost and the value of this minimum. Show indeed it is a minimum.

$$AC' = \frac{-20}{q^2} + 5 = \frac{5q^2 - 20}{q^2}$$

6

$$AC' = 0 \Rightarrow 5q^2 - 20 = 0 \Rightarrow q^2 = 4 \Rightarrow q = -2 \text{ (rejected)} \\ \text{or } q = 2$$

$$AC'' = 40/q^3 \text{ and } AC''(2) = 40/8 = 5 > 0 \Rightarrow q = 2$$

minimizes AC and the value of the minimum is

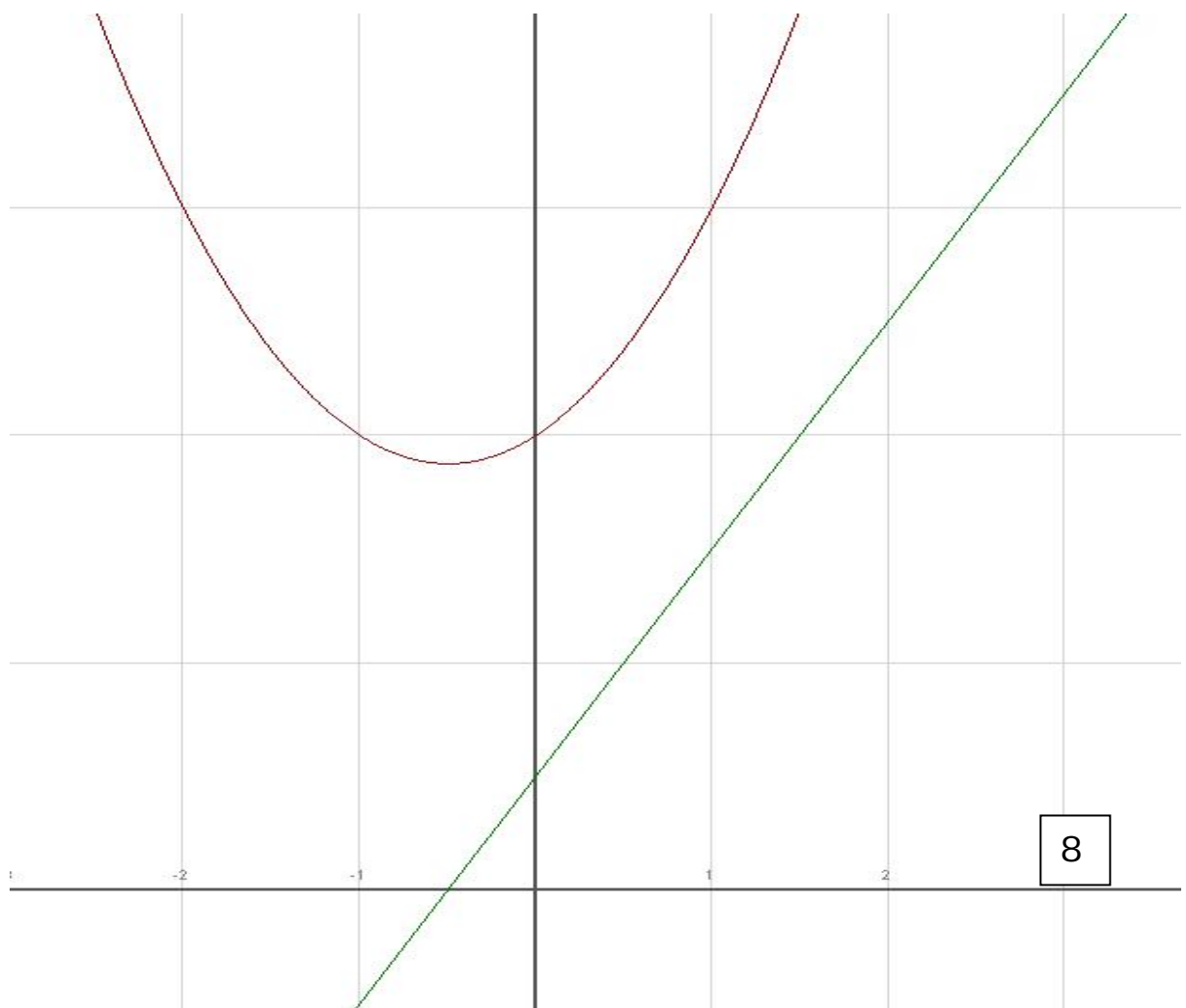
$$AC(2) = 10 + 5 + 10 = 25$$

- (iii) Verify that when $q = 2$, the marginal cost MC equals the Average cost.

2

$$AC(2) = 25, \quad MC(2) = 5 + 20 = 25 \therefore AC = MC \text{ at } q = 2$$

- (iv) Sketch the graphs of the total cost TC and the marginal cost MC functions on the same system of axes.



8