



November 27th, 2007

Unit: 05a – Mathematics 1

GROUP(A)-VERSION A

This paper is not to be removed from the Examination Halls

Student Name :

Student Number :

Tuesday 27th November 13 : 30 pm – 15 : 30 pm

Candidates should answer **NINE** of the following **ELEVEN** questions: **SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

PLEASE TURN OVER

SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The supply and the demand functions for a good are given, respectively, by $q = 4p^2 + 2p - 1$, $q = -4p^2 - 18p + 27$ where p is the price.

Sketch the supply and the demand functions for $p \geq 0$

Determine the equilibrium price and quantity.

Supply $q = 4p^2 + 2p - 1$

(1) It has U shape since it has positive p^2 term

(2) Intercepts: p-intercepts : $q = 0$, $4p^2 + 2p - 1 = 0$,

$$p = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{4 \times 5}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

The points are : $\left(\frac{-1 - \sqrt{5}}{4}, 0\right)$ and $\left(\frac{-1 + \sqrt{5}}{4}, 0\right)$

q-intercept: $p = 0 \Rightarrow q = -1$; $(0, -1)$

(3) The minimum : $q' = 8p + 2 = 0 \Rightarrow p = -1/4$

$\Rightarrow q = 4(-1/4)^2 + 2(-1/4) - 1 = -5/4$; $(-1/4, -5/4)$

$$\text{OR, } p = \frac{-b}{2a} = \frac{-2}{8} = \frac{-1}{4} \Rightarrow q = -5/4 \Rightarrow V(-1/4, -5/4)$$

Demand $q = -4p^2 - 18p + 27$

(1) It has \cap shape since it has negative p^2 term

(2) Intercepts: p-intercepts : $q = 0$, $-4p^2 - 18p + 27 = 0$,

$$p = \frac{9 \pm \sqrt{189}}{-4} = \frac{-9 \pm \sqrt{189}}{4}$$

The points are : $\left(\frac{-9 - \sqrt{189}}{4}, 0\right)$, $\left(\frac{-9 + \sqrt{189}}{4}, 0\right)$

q-intercept: $p = 0 \Rightarrow q = 27$; $(0, 27)$

(3) The minimum : $q' = -8p - 18 = 0 \Rightarrow p = -9/4$

$\Rightarrow q = -4(-9/4)^2 - 18(-9/4) + 27 = 189/4$; $(-9/4, 189/4)$

$$\text{OR, } p = \frac{-b}{2a} = \frac{18}{-8} = -\frac{9}{4} \Rightarrow q = 189/4 \Rightarrow V(-9/4, 189/4)$$

Equilibrium price and quantity

$$4p^2 + 2p - 1 = -4p^2 - 18p + 27 \Rightarrow 8p^2 + 20p - 28 = 0$$

$$\Rightarrow 2p^2 + 5p - 7 = 0 \Rightarrow p = -7/2 \text{ which is economically not}$$

$$\text{feasible, } p = 1 \Rightarrow q = 4(1)^2 + 2(1) - 1 = 5$$

Sketch of the graphs 5 marks each .

4

2. The function $f(x)$ is defined for $x > 0$ by:

$$f(x) = 2\ln(5x) - x^2$$

Find the stationary points of $f(x)$ and specify their nature.

$$f'(x) = \frac{2}{x} - 2x = 0 \Rightarrow 2 - 2x^2 = 0 \Rightarrow x = \pm 1 \quad \boxed{2}$$

$x = -1$ rejected since $x > 0$

$$x = 1 \Rightarrow f(x) = 2\ln 5 - 1, \text{ critical point } (1, 2\ln 5 - 1) \quad \boxed{1}$$

$$f''(x) = \frac{-2}{x^2} - 2 < 0 \text{ for every real } x, x = 1 \text{ maximises } f(x) \quad \boxed{2}$$

3. Determine the following integrals

$$\int (\ln x)^2 dx, \text{ integration by parts}$$

$$u = (\ln x)^2, du = \frac{2 \ln x}{x}, dv = dx \Rightarrow v = x$$

$$\begin{aligned} \int (\ln x)^2 dx &= \int u dv = uv - \int v du = x(\ln x)^2 - 2 \int \ln x dx \\ &= x(\ln x)^2 - 2(x \ln x - x) + C \quad (\int \ln x dx = x \ln x - x + C, \text{ By Parts}) \quad \boxed{5} \end{aligned}$$

$$\int \frac{x^5}{\sqrt{2x^2 + 3}} dx, u^2 = 2x^2 + 3, 2u du = 4x dx, x dx = \frac{1}{2} u du$$

$$= \int \frac{x^4}{\sqrt{2x^2 + 3}} x dx = \frac{1}{2} \int \frac{x^4}{u} u du = \frac{1}{2} \int x^4 du$$

$$\text{But } u^2 = 2x^2 + 3 \Rightarrow x^2 = \frac{u^2 - 3}{2} \Rightarrow x^4 = \frac{(u^2 - 3)^2}{4}$$

$$\frac{1}{2} \int x^4 du = \frac{1}{8} \int (u^2 - 3)^2 du = \frac{1}{8} \int (u^4 - 6u^2 + 9) du = \quad \boxed{5}$$

$$\frac{u^5}{40} - 2u^3 + 9u + C =$$

$$\frac{(\sqrt{2x^2 + 3})^5}{40} - 2(\sqrt{2x^2 + 3})^3 + 9\sqrt{2x^2 + 3} + C$$

4. A firm has marginal cost qe^{q^2} and fixed costs are 5.
Find its total cost function.

$$TC = \int MCdq = \int qe^{q^2} dq, \quad t = q^2, \quad dt = 2q dq$$

$$= \frac{1}{2} \int e^t dt = \frac{1}{2} e^t + C = \frac{1}{2} e^{q^2} + C$$

$$FC = TC(0) \Rightarrow \frac{1}{2} e^0 + C = 5 \Rightarrow \frac{1}{2} + C = 5, \quad C = 9/2$$

$$TC = \frac{1}{2} e^{q^2} + 9/2$$

6

5. The inverse demand function for a good takes the form

$$p = \frac{a}{q+3} \text{ where } a > 3 \text{ some fixed number.}$$

The supply function is $p = q - 1$

Find expressions, in terms of a for the equilibrium price and quantity.

$$p = p \Rightarrow \frac{a}{q+3} = q - 1 \Rightarrow (q-1)(q+3) = a$$

$$\Rightarrow q^2 + 2q - 3 - a = 0$$

$$q = \frac{-2 \pm \sqrt{4 - 4(1)(-3-a)}}{2} = \frac{-2 \pm \sqrt{4 + 4(3+a)}}{2}$$

$$\text{since } a > 3 \Rightarrow a + 3 > 6 \text{ i.e. } a + 3 > 0$$

i.e. q exists

$$q = \frac{-2 \pm \sqrt{4(1+3+a)}}{2} = \frac{-2 \pm 2\sqrt{4+a}}{2} = -1 \pm \sqrt{4+a}$$

$$q = -1 - \sqrt{4+a} \text{ not economically feasible,}$$

$$q = -1 + \sqrt{4+a}, \quad p = q - 1 = -2 + \sqrt{4+a}$$

7

6. A firm's demand function is

$$p = mq + n \quad (m < 0; n > 0)$$

fixed costs are c and variable costs are v per unit.

$$\text{Show that the profit is maximized when } q = \frac{v-n}{2m}$$

The profit $\pi = TR - TC$

$$TR = p \times q \text{ and } TC = VC + FC = vq + c$$

$$\text{Now } p = mq + n \Rightarrow TR = (mq + n) \times q = mq^2 + nq$$

$$\pi = TR - TC = mq^2 + nq - vq - c = mq^2 + (n-v)q - c$$

$$\pi' = 2mq + n - v = 0 \Rightarrow q = \frac{v-n}{2m}$$

$$\pi'' = 2a < 0 \quad (\text{since } a < 0) \Rightarrow q = \frac{v-n}{2m} \text{ maximises the profit.}$$

6

7. Determine the following integrals

$$\int \frac{x-3}{x^2-6x+13} dx \quad u = x^2 - 6x + 13, \quad du = 2(x-3)dx$$

$$= \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 - 6x + 13| + C$$

5

$$\int \frac{dx}{\cos^2 x (\tan^2 x + 6 \tan x + 8)}, \quad u = \tan x, \quad du = dx/\cos^2 x$$

$$= \int \frac{du}{u^2 + 6u + 8}, \quad \frac{1}{u^2 + 6u + 8} = \frac{a}{u+2} + \frac{b}{u+4}$$

$$1 = a(u+4) + b(u+2) \Rightarrow a = \frac{1}{2}, \quad b = -\frac{1}{2}$$

$$\int \frac{du}{u^2 + 6u + 8} = \frac{1}{2} \ln |u+2| - \frac{1}{2} \ln |u+4| + C = \frac{1}{2} \ln \left| \frac{u+2}{u+4} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\tan x + 2}{\tan x + 4} \right| + C$$

5

SECTION B

Answer **TWO** questions from this section (20 marks each)

8.(a) A firm is a monopoly for the good it produces, It has a marginal cost function $MC = 12q^2 + 16$ and fixed costs of 40. The demand equation for its good is given by $P + 4q = 80$ where p is the price. Find expressions in terms of q , for the total revenue and profit. Determine the value of q that maximises the profit.

$$TR = pxq, \text{ but } p + 4q = 80 \Rightarrow p = 80 - 4q$$

$$TR = (80 - 4q)xq = 80q - 4q^2$$

$$TC = \int MC dq = \int (12q^2 + 16) dq = 4q^3 + 16q + C$$

$$FC = TC(0) = 40 \Rightarrow 0 + 0 + C = 40, \quad C = 40$$

$$TC = 4q^3 + 16q + 40$$

$$\pi = TR - TC = 80q - 4q^2 - 4q^3 - 16q - 40 = -4q^3 - 4q^2 + 64q - 40$$

$$\pi' = -12q^2 - 8q + 64 = 0 \Rightarrow -3q^2 - 2q + 16 = 0$$

$$q = \frac{2 \pm \sqrt{4 - 4(-3)(16)}}{-6} = \frac{2 \pm \sqrt{4 + 192}}{-6} = \frac{2 \pm \sqrt{196}}{-6}$$

$$q = \frac{2 \pm 14}{-6} \Rightarrow q = 2, \quad q = -16/6 = -8/3 \text{ which is economically}$$

not feasible, therefore $q = 2$

$$\pi'' = -12q - 4, \quad \pi''(2) = -12(2) - 4 = -28 < 0 \Rightarrow q = 2$$

maximises the profit.

(b) Determine the following integrals

$$\int \frac{dx}{x^{1/2} - x^{3/2}} = \int \frac{dx}{x^{1/2}(1-x)} = \int \frac{dx}{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x})}$$

$$u = \sqrt{x} \Rightarrow du = \frac{dx}{2\sqrt{x}}$$

$$= 2 \int \frac{du}{(1-u)(1+u)}, \frac{1}{(1-u)(1+u)} = \frac{a}{1-u} + \frac{b}{1+u}$$

$$1 = a(1-u) + b(1+u) \Rightarrow a = \frac{1}{2}, b = \frac{1}{2}$$

$$2 \int \frac{du}{(1-u)(1+u)} = 2 \left[-\frac{1}{2} \ln|1-u| + \frac{1}{2} \ln|1+u| \right] + C$$

$$2 \left[\frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| \right] + C = \ln \left| \frac{1+u}{1-u} \right| + C = \ln \left| \frac{1+\sqrt{x}}{1-\sqrt{x}} \right| + C \quad \boxed{5}$$

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{dx}{e^x - \frac{1}{e^x}} = \int \frac{e^x dx}{e^{2x} - 1}, u = e^x, du = e^x dx$$

$$= \int \frac{du}{u^2 - 1}, \frac{1}{(u-1)(u+1)} = \frac{a}{u-1} + \frac{b}{u+1}$$

$$1 = a(u-1) + b(u+1) \Rightarrow a = -\frac{1}{2}, b = \frac{1}{2}$$

$$\int \frac{du}{u^2 - 1} = -\frac{1}{2} \ln|u-1| + \frac{1}{2} \ln|u+1| + C = \frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| + C \quad \boxed{5}$$

$$= \frac{1}{2} \ln \left| \frac{e^x + 1}{e^x - 1} \right| + C$$

9. (a) A monopoly has fixed costs of **10** and Average variable cost function **q^2+4** . the demand equation for its product is **$p+q=20$** . Determine the profit function in terms of q . Determine also the production level that maximises the profit.

$$TR = pxq, \text{ but } p + q = 20 \Rightarrow p = 20 - q$$

$$TR = (20 - q)xq = 20q - q^2$$

$$TC = VC + FC = q \times AVC + FC = q^3 + 4q + 10$$

$$\pi = TR - TC = 20q - q^2 - q^3 - 4q - 10 = -q^3 - q^2 + 16q - 10$$

$$\pi' = -3q^2 - 2q + 16 = 0 \Rightarrow$$

2

2

2

$$q = \frac{2 \pm \sqrt{4 - 4(-3)(16)}}{-6} = \frac{2 \pm \sqrt{4 + 192}}{-6} = \frac{2 \pm \sqrt{196}}{-6}$$

$$q = \frac{2 \pm 14}{-6} \Rightarrow q = 2, q = -16/6 = -8/3 \text{ which is economically } \boxed{2}$$

not feasible, therefore $q = 2$

$$\pi'' = -12q - 4, \pi''(2) = -12(2) - 4 = -28 < 0 \Rightarrow q = 2 \text{ maximises the profit. } \boxed{2}$$

(b) At which values of x will the function

$$f(x) = x^7/7 + x^6/6 - 2x^5/5$$

has a local maximum? Explain your answer.

$$f'(x) = x^6 + x^5 - 2x^4 = 0 \Rightarrow x^4(x^2 + x - 2) = 0$$

$$x = 0, x = 1, x = -2$$

$$f''(x) = 6x^5 + 5x^4 - 8x^3$$

at $x = 1, f''(1) = 3 > 0$ minimum

at $x = -2, f''(-2) = -48 < 0$ maximum

at $x = 0, f''(0) = 0$ test fails

the sign of $f'(x)$:

-2	0	1
+	-	+
↙	↘	↙

At $x = 0, f$ is monotone, $x = 0$ is an inflection point

10.(a) A firm's marginal revenue function is $MR = 11 - q$

The firm's marginal cost function is

$$MC = q^2 - 3q + 3$$

where q is either the quantity sold or produced. Find the profit-maximizing level of output and verify that it is a maximum.

$$TC = \int (q^2 - 3q + 3) dq = q^3/3 - 3q^2/2 + 3q + C \quad \boxed{2}$$

$$TR = \int (11 - q) dq = 11q - q^2/2 \quad \boxed{2}$$

$$\pi = TR - TC = 11q - q^2/2 - q^3/3 + 3q^2/2 - 3q - C \quad \boxed{2}$$

$$\pi' = -q^2/3 + q^2 + 8q - C \Rightarrow \pi' = -q^2 + 2q + 8 = 0$$

$$q = -2 \text{ which is economically not feasible, } q = 4$$

$$\pi'' = -2q + 2 = -2(4) + 2 = -6 < 0, q = 4 \text{ maximises the profit } \boxed{4}$$

Another method

$$MR = MC : 11 - q = q^2 - 3q + 3 \Rightarrow -q^2 + 2q + 8 = 0$$

$$q = -2 \text{ which economically not feasible, } q = 4$$

(b) Determine the following integrals

$$\int \frac{\ln x}{\sqrt{x}} dx = \int x^{-1/2} \ln x dx, \text{ integration by parts}$$

$$u = \ln x, du = dx/x, dv = x^{-1/2} dx \Rightarrow v = 2x^{1/2}$$

$$\int \frac{\ln x}{\sqrt{x}} dx = \int u dv = uv - \int v du = 2\sqrt{x} \ln x - 2 \int \frac{x^{1/2}}{x} dx \quad \boxed{5}$$

$$= 2\sqrt{x} \ln x - 2 \int x^{-1/2} dx = 2\sqrt{x} \ln x - 4\sqrt{x} + C$$

$$\int \frac{\sqrt{x-1}}{x-2} dx, u^2 = x-1, 2u du = dx$$

$$= 2 \int \frac{u^2}{x-2} du, \text{ But } u = x-1 \Rightarrow x = u+1$$

$$= 2 \int \frac{u^2}{u+1-2} du = 2 \int \frac{u^2}{u-1} du = 2 \int \frac{u^2-1+1}{u-1} du$$

$$= 2 \int \left(u+1 + \frac{1}{u-1} \right) du = u^2 + 2u + 2 \ln|u-1| + C$$

$$= x-1 + 2\sqrt{x-1} + 2 \ln|\sqrt{x-1}| + C$$

5

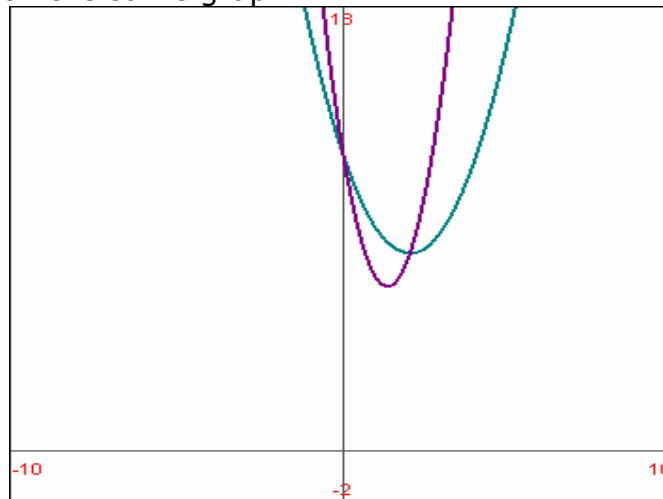
11. A firm faces a total cost function $TC = 2q^3 - 8q^2 + 24q$

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

$$AC = \frac{TC}{q} = 2q^2 - 8q + 24, MC = (TC)' = 6q^2 - 16q + 24$$

2

(ii) Sketch the average cost (AC) and the marginal cost (MC) on the same graph.



10

(iii) If price is \$ 15, which level of output will a profit maximising firm choose?

$$TR = 15q$$

$$\pi = TR - TC = 15q - 2q^3 + 8q^2 - 24q = -2q^3 + 8q^2 - 9q$$

$$\pi' = -6q^2 + 16q - 9 = 0 \Rightarrow q = 1, q = 3/2$$

$\pi'' = -12q + 16 = -12(1) + 16 = 4 > 0 \Rightarrow q = 1$ minimises the profit

$\pi''(3/2) = -12(3/2) + 16 = -2 < 0 \Rightarrow q = 3/2$ maximises the profit.

8

END OF PAPER