International Institute for
Technology and Management

November 27th, 2007 Unit: 05a – Mathematics 1

GROUP(A)-VERSION A

This paper is not to be removed from the Examination Halls

Student Name :

Student Number :

Tuesday 27th November 13 : 30 pm – 15 : 30 pm

Candidates should answer **NINE** of the following **ELEVEN** questions:**SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

PLEASE TURN OVER

SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The supply and the demand functions for a good are given ,respectively, by **$q = 4p^2 + 2p - 1$, $q = -4p^2 - 18p + 27$** where p is the price.

Sketch the supply and the demand functions for **p** ≥ 0

Determine the equilibrium price and quantity.

Supply **$q = 4p^2 + 2p - 1$**

(1)It has U shape since it has positive p^2 term

(2)Intercepts: <u>p-intercepts</u> : q = 0 , $4p^2 + 2p - 1 = 0$,

$$p = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{4 \times 5}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

The points are : $\left(\frac{-1-\sqrt{5}}{4}, 0\right)$ and $\left(\frac{-1+\sqrt{5}}{4}, 0\right)$

<u>q-intercpt</u>: p = 0 $\Rightarrow$ q = -1 ; (0,-1)

(3) The minimum : q' = 8p + 2 = 0 $\Rightarrow$ p = -1/4

$\Rightarrow$ $q = 4(-1/4)^2 + 2(-1/4) - 1 = -5/4$; (-1/4 , -5/4)

OR, $p = \frac{-b}{2a} = \frac{-2}{8} = \frac{-1}{4} \Rightarrow$ q =-5/4 $\Rightarrow$ V(-1/4 , -5/4)

Demand **$q = -4p^2 - 18p + 27$**

(1)It has $\cap$ shape since it has negative p^2 term

(2)Intercepts: <u>p-intercepts</u> : q = 0 , $-4p^2 - 18p + 27 = 0$,

$$p = \frac{9 \pm \sqrt{189}}{-4} = \frac{-9 \pm \sqrt{189}}{4}$$

The points are : $(\frac{-9-\sqrt{189}}{4}, 0)$, $(\frac{-9+\sqrt{189}}{4}, 0)$

<u>q-intercpt</u>: p = 0 $\Rightarrow$ q = 27 ; (0,27)

(3) The minimum : q' = -8p - 18 = 0 $\Rightarrow$ p = -9/4

$\Rightarrow$ $q = -4(-9/4)^2 - 18(-9/4) + 27 = 189/4$; (-9/4 , 189/4)

OR, $p = \frac{-b}{2a} = \frac{18}{-8} = -\frac{9}{4} \Rightarrow$ q =-5/4 $\Rightarrow$ V(-9/4 , 189/4)

Equilibrium price and quantity

$4p^2 + 2p - 1 = -4p^2 - 18p + 27$ $\Rightarrow$ $8p^2 + 20p - 28 = 0$

$\Rightarrow$ $2p^2 + 5p - 7 = 0$ $\Rightarrow$ p = -7/2 which is economically not feasible, p = 1 $\Rightarrow$ $q = 4(1)^2 + 2(1) - 1 = 5$

4

Sketch of the graphs 5 marks each .

2. The function f(x) is defined for x > 0 by:

$$f(x) = 2\ln(5x) - x^2$$

Find the stationary points of f(x) and specify their nature.

2

$f'(x) = \frac{2}{x} - 2x = 0 \Rightarrow 2 - 2x^2 = 0 \Rightarrow x = \pm 1$ 2

x = - 1 rejected since x > 0

$x = 1 \Rightarrow f(x) = 2\ln 5 - 1$, critical point (1, 2ln5 – 1) 1

$f''(x) = \frac{-2}{x^2} - 2 \prec 0$ for every real x , x = 1 maximises f(x) 2

3. Determine the following integrals

$\int (\ln x)^2 dx$, integration by parts

$u = (\ln x)^2$, $du = \frac{2\ln x}{x}$, $dv = dx \Rightarrow v = x$

$\int (\ln x)^2 dx = \int u dv = uv - \int v du = x(\ln x)^2 - 2\int \ln x dx$

$= x(\ln x)^2 - 2(x\ln x - x) + C$ ($\int \ln x dx = x \ln x - x + c$,By Parts) 5

$\int \frac{x^5}{\sqrt{2x^2+3}} dx$, $u^2 = 2x^2 + 3$, $2udu = 4xdx$, $xdx = ½ udu$

$= \int \frac{x^4}{\sqrt{2x^2+3}} xdx = ½ \int \frac{x^4}{u} udu = ½ \int x^4 du$

But $u^2 = 2x^2 + 3 \Rightarrow x^2 = \frac{u^2-3}{2} \Rightarrow x^4 = \frac{(u^2-3)^2}{4}$

$½ \int x^4 du = \frac{1}{8}\int (u^2-3)^2 du = \frac{1}{8}\int (u^4 - 6u^2 + 9) du =$ 5

$\frac{u^5}{40} - 2u^3 + 9u + C =$

$$\frac{(\sqrt{2x^2+3})^5}{40} - 2(\sqrt{2x^2+3})^3 + 9\sqrt{2x^2+3} + C$$

4. A firm has marginal cost qe^{q^2} and fixed costs are 5.

Find its total cost function.

TC = $\int MC dq = \int q e^{q^2} dq$, t = q^2 , dt = 2q dq

= ½ $\int e^t dt$ = ½ e^t + C = ½ e^{q^2} + C

FC = TC(0) $\Rightarrow$ ½ e^0 + C = 5 $\Rightarrow$ ½ + C = 5 , C = 9/2

TC = ½ e^{q^2} + 9/2 **6**

5. The inverse demand function for a good takes the form

$p = \frac{a}{q+3}$ where a > 3 some fixed number.

The supply function is $p = q - 1$

Find expressions,in terms of a for the equilibrium price and quantity.

p = p $\Rightarrow \frac{a}{q+3} = q - 1 \Rightarrow$ (q-1)(q+3) = a

$\Rightarrow q^2 + 2q - 3 - a = 0$

q = $\frac{-2 \pm \sqrt{4 - 4(1)(-3-a)}}{2} = \frac{-2 \pm \sqrt{4 + 4(3+a)}}{2}$

since a > 3 $\Rightarrow$ a +3 > 6 i.e. a + 3 > 0

i.e. q exists

q = $\frac{-2 \pm \sqrt{4(1+3+a)}}{2} = \frac{-2 \pm 2\sqrt{4+a}}{2} = -1 \pm \sqrt{4+a}$

q= -1 - $\sqrt{4+a}$ not economically feasible , **7**

q = -1 + $\sqrt{4+a}$, p = q – 1 = -2 + $\sqrt{4+a}$

6. A firm's demand function is

p = mq + n (m < 0 ; n > 0)

fixed costs are ***c*** and variable costs are ***v*** per unit.

Show that the profit is maximized when q = $\frac{v-n}{2m}$

The profit π = TR – TC

TR = p x q and TC = VC + FC = ***vq + c***

Now p = mq + n $\Rightarrow$ TR = (mq + n) x q= mq^2 + nq

π = TR – TC = mq^2 + nq – vq – c = mq^2 + (n– v)q – c

π' = 2mq + n – v = 0 $\Rightarrow$ q = $\frac{v-n}{2m}$

π'' = 2a < 0 (since a < 0) $\Rightarrow$ q = $\frac{v-n}{2m}$ maximises the profit. **6**

7. Determine the following integrals

$$\int \frac{x-3}{x^2-6x+13}dx \quad u = x^2 - 6x + 13 \text{ , } du = 2(x-3)dx$$

$$= ½ \int \frac{du}{u} = ½ \ln |u| + C = ½ \ln | x^2 - 6x + 13| + C$$ 5

$$\int \frac{dx}{\cos^2 x(\tan^2 x + 6\tan x + 8)} \text{ , } u = \tan x \text{ , } du = dx/\cos^2 x$$

$$= \int \frac{du}{u^2+6u+8} \text{ , } \frac{1}{u^2+6u+8} = \frac{a}{u+2} + \frac{b}{u+4}$$

$1 = a(u+4) + b(u+2) \Rightarrow a = ½ \text{ , } b = - ½$

$$\int \frac{du}{u^2+6u+8} = ½ \ln |u+2| - ½ \ln|u+4| + C = ½ \ln\left|\frac{u+2}{u+4}\right| + C$$

$$= ½ \ln\left|\frac{\tan x + 2}{\tan x + 4}\right| + C$$ 5

SECTION B

Answer **TWO** questions from this section (20 marks each)

8.(a) A firm is a monopoly for the good it produces, It has a marginal cost function MC = $12q^2 + 16$ and fixed costs of 40.The demand equation for its good is given by P + 4q = 80 where p is the price. Find expressions in terms of q, for the total revenue and profit. Determine the value of q that maximises the profit.

TR = pxq , but p + 4q = 80 $\Rightarrow$ p = 80 – 4q 2

TR = (80 -4q)xq = $80q - 4q^2$

TC = $\int MCdq = \int (12q^2 + 16)dq = 4q^3 + 16q + C$ 2

FC=TC(0) = 40 $\Rightarrow$ 0 + 0 + C = 40 , C = 40

TC = $4q^3 + 16q + 40$

π = TR – TC = $80q - 4q^2 - 4q^3 - 16q - 40 = - 4q^3 - 4q^2 + 64q - 40$ 2

$\pi' = -12q^2 - 8q + 64 = 0 \Rightarrow -3q^2 - 2q + 16 = 0$

$$q = \frac{2 \pm \sqrt{4-4(-3)(16)}}{-6} = \frac{2 \pm \sqrt{4+192}}{-6} = \frac{2 \pm \sqrt{196}}{-6}$$

$q = \frac{2 \pm 14}{-6} \Rightarrow$ q = 2 , q = - 16/6 = - 8/3 which is economically not feasible , therefore q = 2 2

$\pi'' = -12q - 4$, $\pi''(2) = -12(2) - 4 = - 28 < 0 \Rightarrow$ q = 2 maximises the profit. 2

(b) Determine the following integrals

$$\int \frac{dx}{x^{1/2} - x^{3/2}} = \int \frac{dx}{x^{1/2}(1-x)} = \int \frac{dx}{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x})}$$

u = $\sqrt{x} \Rightarrow$ du = $\frac{dx}{2\sqrt{x}}$

= 2 $\int \frac{du}{(1-u)(1+u)}$, $\frac{1}{(1-u)(1+u)} = \frac{a}{1-u} + \frac{b}{1+u}$

1 = a(1- u) + b(1+u) $\Rightarrow$ a = ½ , b = ½

$2\int \frac{du}{(1-u)(1+u)}$ = 2 [- ½ ln|1-u| + ½ ln|1+u|] + C

2 [½ $\ln\left|\frac{1+u}{1-u}\right|$] + C = $\ln\left|\frac{1+u}{1-u}\right|$ + C = $\ln\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right|$ + C 5

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{dx}{e^x - \frac{1}{e^x}} = \int \frac{e^x dx}{e^{2x} - 1}$$, u = e^x , du = e^x dx

= $\int \frac{du}{u^2 - 1}$, $\frac{1}{(u-1)(u+1)} = \frac{a}{u-1} + \frac{b}{u+1}$

1 = a(u - 1) + b(u+1) $\Rightarrow$ a = - ½ , b = ½

$\int \frac{du}{u^2 - 1}$ = -½ ln|u-1| + ½ ln|u+1| + C = ½ $\ln\left|\frac{u+1}{u-1}\right| + C$ 5

= ½ $\ln\left|\frac{e^x + 1}{e^x - 1}\right| + C$

9. (a) A monopoly has fixed costs of **10** and Average variable cost function **q^2+4** .the demand equation for its product is **p+q =20**.Determine the profit function in terms of q. Determine also the production level that maximises the profit.

TR = pxq , but p + ***q = 20*** $\Rightarrow$ p = 20 – q 2

TR = (20 - q)xq = 20q – q^2 2

TC = VC + FC = q x AVC + FC = q^3 + 4q+ 10

π = TR – TC = 20q – q^2 -q^3 - 4q-10 = -q^3 -q^2 +16q -10 2

π' = -3q^2-2q +16 = 0 $\Rightarrow$

$q = \dfrac{2 \pm \sqrt{4-4(-3)(16)}}{-6} = \dfrac{2 \pm \sqrt{4+192}}{-6} = \dfrac{2 \pm \sqrt{196}}{-6}$

$q = \dfrac{2 \pm 14}{-6} \Rightarrow q = 2$, $q = -16/6 = -8/3$ which is economically **2**

not feasible , therefore $q = 2$ **2**

$\pi'' = -12q - 4$, $\pi''(2) = -12(2) - 4 = -28 < 0 \Rightarrow q = 2$ maximises the profit.

(b) At which values of x will the function

$f(x) = x^7/7 + x^6/6 - 2x^5/5$

has a local maximum? Explain your answer. **2**

$f'(x) = x^6 + x^5 - 2x^4 = 0 \Rightarrow x^4(x^2 + x - 2) = 0$ **2**

$x = 0$, $x = 1$, $x = -2$

$f''(x) = 6x^5 + 5x^4 - 8x^3$ **2**

at $x = 1$, $f''(1) = 3 > 0$ minimum **2**

at $x = -2$, $f''(-2) = -48 < 0$ maximum **2**

at $x = 0$, $f''(0) = 0$ test fails

the sign of $f'(x)$:

	-2		0		1	
+		-	- -	-		+
↗		↘		↘		↗

At x= 0 , f is monotone , x =0 is an inflection point

10.(a) A firm's marginal revenue function is $MR = 11 - q$

The firm's marginal cost function is

$MC = q^2 - 3q + 3$

where q is either the quantity sold or produced. Find the profit-maximizing level of output and verify that it is a maximum.

$TC = \int (q^2 - 3q + 3)\,dq = q^3/3 - 3q^2/2 + 3q + C$ **2**

$TR = \int (11 - q)\,dq = 11q - q^2/2$ **2**

$\pi = TR - TC = 11q - q^2/2 - q^3/3 + 3q^2/2 - 3q - C$

$\pi = -q^3/3 + q^2 + 8q - C \Rightarrow \pi' = -q^2 + 2q + 8 = 0$ **2**

$q = -2$ which is economically not feasible, $q = 4$

$\pi'' = -2q + 2 = -2(4) + 2 = -6 < 0$, $q = 4$ maximises the profit **4**

<u>Another method</u>

MR = MC : $11 - q = q^2 - 3q + 3 \Rightarrow -q^2 + 2q + 8 = 0$

$q = -2$ which economically not feasible, $q = 4$

(b) Determine the following integrals

$\int \dfrac{\ln x}{\sqrt{x}}\,dx = \int x^{-1/2} \ln x\,dx$, integration by parts

$u = \ln x$, $du = dx/x$, $dv = x^{-1/2}\,dx \Rightarrow v = 2x^{1/2}$

$\int \dfrac{\ln x}{\sqrt{x}}\,dx = \int u\,dv = uv - \int v\,du = 2\sqrt{x}\ln x - 2\int \dfrac{x^{1/2}}{x}\,dx$ **5**

$= 2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$

$$\int \frac{\sqrt{x-1}}{x-2}\,dx$$, $u^2 = x - 1$, $2u\,du = dx$

$$= 2\int \frac{u^2}{x-2}\,du$$, But $u = x - 1 \Rightarrow x = u + 1$

$$= 2\int \frac{u^2}{u+1-2}\,du = 2\int \frac{u^2}{u-1}\,du = 2\int \frac{u^2-1+1}{u-1}\,du$$

$$= 2\int (u+1+\frac{1}{u-1})du = u^2 + 2u + 2\ln|u-1| + C$$

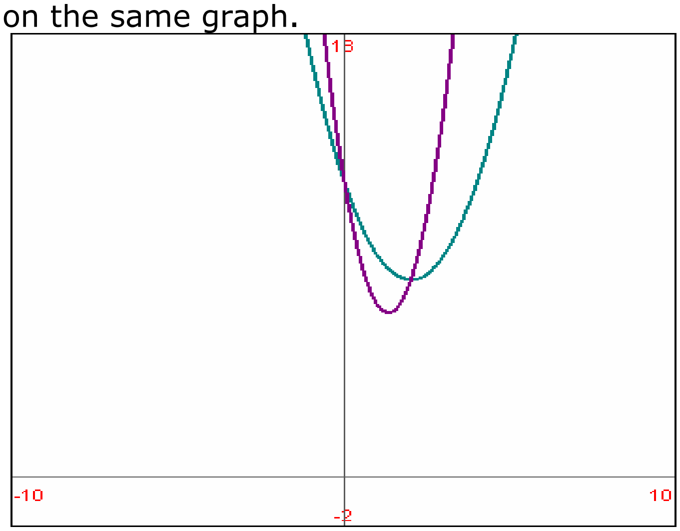
$= x - 1 + 2\sqrt{x-1} + 2\ln|\sqrt{x-1}| + C$ 5

11. A firm faces a total cost function $TC = 2q^3 - 8q^2 + 24q$

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

$AC = \frac{TC}{q} = 2q^2 - 8q + 24$, $MC = (TC)' = 6q^2 - 16q + 24$ 2

(ii) Sketch the average cost (AC) and the marginal cost(MC) on the same graph.



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(iii) If price is \$ 15 , which level of output will a profit maximising firm choose?

TR = 7.5q

$\pi = TR - TC = 15q - 2q^3 + 8q^2 - 24q = -2q^3 + 8q^2 - 9q$ 8

$\pi' = -6q^2 + 16q - 9 = 0 \Rightarrow q = 1 , q = 3/2$

$\pi'' = -12q + 16 = -12(1) + 16 = 4 > 0 \Rightarrow q = 1$ minimises the profit

$\pi''(3/2) = -12(3/2) + 16 = -2 < 0 \Rightarrow q = 3/2$ maximises the profit.

END OF PAPER