**International Institute for
Technology and Management**

November 2008 **Unit: 05a – Mathematics 1**

GROUP(A)-VERSION A

This paper is not to be removed from the Examination Halls

Student Name :

Student Number :

TIME ALLOWED: 2 hours

Candidates should answer **NINE** of the following **ELEVEN** questions:**SEVEN** from section A (60 marks in total) and **TWO** from section B (20 marks each).

Candidates are strongly advised to divide their time accordingly.

Graph paper is provided at the student request.

Calculators **May NOT** be used for this paper.

PLEASE TURN OVER

SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. The functions f(x) and g(x) are:

$f(x) = 2x^2 + x - 10$, $g(x) = 7 - 3x^2 - 4x$

Sketch the graphs of f and g, and determine the x-coordinates of their points of intersection.

$f(x) = 2x^2 + x - 10$

-It should be realized that f(x) has a parabolic U shape since it has a positive x^2 term.

-An accurate sketch will need to indicate where the curve cuts the axes:

x-intercepts : y = 0 $\Rightarrow$ $2x^2 + x - 10 = 0$

This can be solved using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(2)(-10)}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{81}}{4} = \frac{-1 \pm 9}{4}$$ and the x-intercepts are:

6

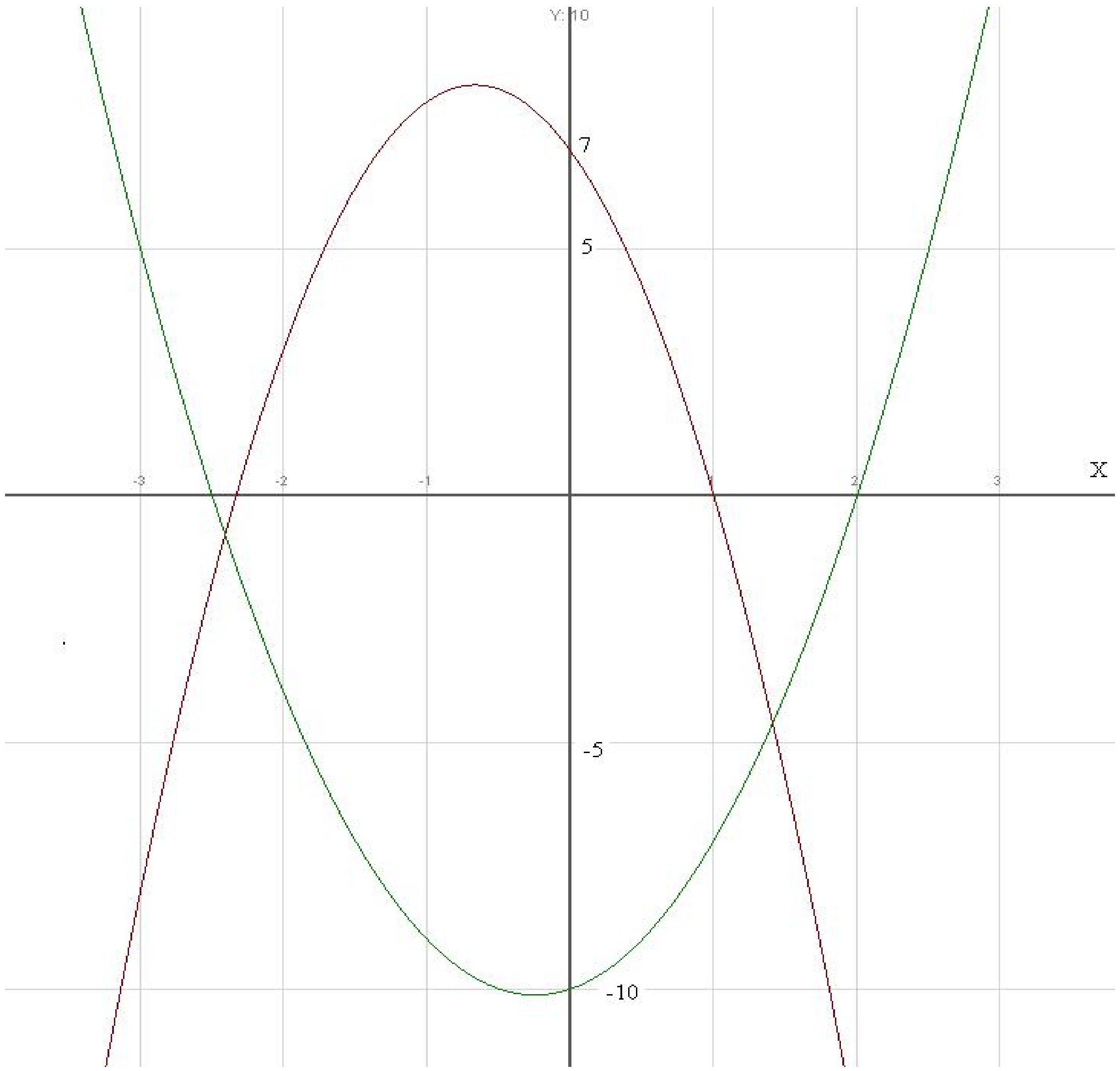
$(-5/2, 0)$ and $(2, 0)$

y-intercept: x = 0 $\Rightarrow$ y = -10 $\therefore$ (0 , - 10)

- An accurate sketch will need to show the minimum of the graph of f(x),we know it's a minimum from the U shape.
  The minimum can be found in one of two ways :
  By differentiation : $f(x) = 2x^2 + x - 10 \Rightarrow f'(x) = 4x + 1 = 0$
  $\Rightarrow$ x = - ¼ , substituting this in f(x) , $y = 2(-1/4)^2 + (-1/4) - 10$
  $= -162/16 = -81/8$
  $\therefore$ V(-1/4 , - 81/8)

  OR by finding the vertex : $x = \frac{-b}{2a} = \frac{-1}{2(2)} = \frac{-1}{4}$

  Now you can Sketch the graph of f(x):
  -You know it has a U shape
  - You know the intercepts with the axes:
  $(-5/2, 0)$ and $(2, 0)$ and (0 , -10)
  -You know the Vertex (minimum) : (-1/4 , - 81/8)



$g(x) = -3x^2 - 4x + 7$

-It should be realized that f(x) has a parabolic ∩ shape since it has a negative x^2 term.

-An accurate sketch will need to indicate where the curve cuts the axes:

<u>x-intercepts</u> : $y = 0 \Rightarrow -3x^2 -4x +7 = 0$

$a + b + c = 0 \Rightarrow x = 1$ or $x = c/a = -7/3$

and the x-intercepts are: $(-7/3,0)$ and $(1,0)$

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<u>y-intercept</u>: $x = 0 \Rightarrow y = 7 \therefore (0, 7)$

- An accurate sketch will need to show the maximum of the graph of f(x),we know it's a maximum from the ∩ shape.
  The maximum can be found in one of two ways :
  By differentiation : $g(x) = -3x^2 -4x +7 \Rightarrow g'(x) = -6x -4 = 0$
  $\Rightarrow x = -2/3$, substituting this in f(x) , $y = -3(-2/3)^2 -4(-2/3) + 7$
  $= 25/3 \therefore V(-2/3, 25/3)$

OR by finding the vertex : $x = \frac{-b}{2a} = \frac{4}{2(-3)} = \frac{-2}{3}$

Now you can Sketch the graph of g(x):

-You know it has a ∩ shape

- You know the intercepts with the axes: $(-7/3,0)$ and $(1,0)$ and (0 , 7)

-You know the Vertex (maximum) : (-2/3 , 25/3)

Intersection points: $2x^2 + x - 10 = -3x^2 - 4x + 7$

$\Rightarrow 5x^2 + 5x - 17 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 - 4(5)(-17)}}{2(5)}$$

$$x = \frac{-5 \pm \sqrt{365}}{10}$$

2

2. Find the maximum value of the function:

$$f(x) = (1+x)e^{\frac{-x}{4}}$$

Show that it is indeed a maximum.

$f'(x) = (1)e^{-x/4} - ¼\,(1+x)\,e^{-x/4} = (1 - ¼ - x/4)\,e^{-x/4}$

$f'(x) = 0 \Rightarrow (¾ - x/4)\,e^{-x/4} = 0$ but $e^{-x/4} \neq 0$

$\Rightarrow ¾ - x/4 = 0 \Rightarrow x = 3$

$f''(x) = -¼\,e^{-x/4} - ¼\,(¾ - x/4)\,e^{-x/4} = (-7/16 + x/16)\,e^{-x/4}$

at x = 3 , $f''(3) = (-7/16 + 3/16)\,e^{-3/4} = -4/16\,e^{-3/4} = -¼\,e^{-3/4} < 0$

$\Rightarrow$ x = 3 maximises f(x).

The maximum is obtained by finding f(3)

$f(3) = 4\,e^{-3/4}$

8

3. Determine the following integrals $\int \frac{\cos(\ln x)}{x}\,dx$

4

Let u = lnx $\Rightarrow$ du = dx/x ,becomes $\int \cos u\,du$ = sin u + C = sin(lnx) + C

$\int x^3\sqrt{4x^2+1}\,dx$ **Let u = $4x^2$ + 1 $\Rightarrow$ du = 8x dx $\Rightarrow$ dx = $\frac{du}{8x}$**

$$\int x^3\sqrt{4x^2+1}\,dx = \int x^3\sqrt{u}\,\frac{du}{8x} = \int x^2\sqrt{u}\,\frac{du}{8} = \frac{1}{8}\int x^2\sqrt{u}\,du$$

But u = $4x^2$ + 1 $\Rightarrow$ x^2 = (u – 1) / 4

$$= \frac{1}{32}\int (u-1)(u)^{\frac{1}{2}}du = \frac{1}{32}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du = \frac{1}{32}\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{u^{\frac{5}{2}}}{80} - \frac{u^{\frac{3}{2}}}{48} + C = \frac{(4x^2+1)^{\frac{5}{2}}}{80} - \frac{(4x^2+1)^{\frac{3}{2}}}{48} + C$$

6

4. A firm has average variable cost

$$AVC = 2q^2 + 5q + \frac{\ln(q^3+2)}{q}$$

and fixed costs of 4 .Find the total cost function and the marginal cost function.

VC = q x AVC = $2q^3 + 5q^2 + \ln(q^3 + 2)$

TC = VC + FC = $2q^3 + 5q^2 + \ln(q^3 + 2) + 4$ 3

MC = $\frac{d}{dq}TC = 6q^2 + 10q + \frac{3q^2}{q^3+2}$ 3

5. The marginal cost is a function of output as follows :

MC = $10 - q + q^2$

Determine the extra cost which is incurred when production is increased from 2 to 4 .

The increase is TC(4) – TC(2) = $\int_2^4 MC\, dq$ 6

$= \int_2^4 (10 - q + q^2)\, dq = 10q - \frac{q^2}{2} + \frac{q^3}{3}\Big|_2^4$

= [10(4) – 8 + 64/3] – [20 – 2 + 8/3] = 160/3 – 62/3 = 98/3

6. Find the positive number a which is such that $\int_1^a \left(1 + \frac{2}{x^2}\right)dx = 2$

$$\int_1^a (1 + 2x^{-2})dx = 2 \Rightarrow \left(x + \frac{2x^{-2+1}}{-2+1}\right)\Big|_1^a = 2 \Rightarrow \left(x - \frac{2}{x}\right)\Big|_1^a = 2$$

$\Rightarrow \left(a - \frac{2}{a}\right) - \left(1 - \frac{2}{1}\right) = 2 \Rightarrow a - \frac{2}{a} + 1 = 2 \Rightarrow a^2 - a - 2 = 0$

6

$\Rightarrow$ a = 2 or a = -1 which is rejected since a > 0 $\Rightarrow$ a = 2

7. Determine the following integrals

$\int \frac{2\sqrt{x}+1}{\sqrt{x}(x+\sqrt{x}-2)}dx$ $u = x + \sqrt{x} - 2 \Rightarrow du = (1 + \frac{1}{2\sqrt{x}})dx$

$du = \frac{2\sqrt{x}+1}{2\sqrt{x}}dx \Rightarrow \frac{2\sqrt{x}+1}{\sqrt{x}}dx = 2du$ the integral becomes

$\int \frac{2du}{u} = 2\ln|u| + C = 2\ln|x + \sqrt{x} - 2| + C$ 5

$$\int \frac{\cos\ xdx}{\sin^2 x + 2\sin\ x + 1} \qquad u = \sin x \Rightarrow du = \cos x\ dx$$

$$= \int \frac{du}{u^2 + 2u + 1} = \int \frac{du}{(u+1)^2} = \int (u+2)^{-2} du$$

$$= \frac{(u+1)^{-1}}{-1} + C = \frac{-1}{u+1} + C = \frac{-1}{\sin x + 1} + C$$

5

SECTION B

Answer **TWO** questions from this section (20 marks each)

8. (a) A monopoly has fixed costs of **10** and marginal cost function $\mathbf{3q^2+4}$ .the demand equation for its product is $\mathbf{p+q=20}$.Determine the profit function in terms of **q**. Determine also the production level that maximises the profit.

(a) TC = $\int MCdq = \int (3q^2 + 4)dq$ = $q^3 + 4q + C$

FC = TC(0) = 10 $\Rightarrow$ 0 + 0 + C = 10 , C = 10

TC = $q^3 + 4q + 10$

TR = pq , but p + q = 20 $\Rightarrow$ p = 20 – q

TR = (20 – q)q = 20q – q^2

π = TR – TC = 20q – q^2 - q^3 - 4q - 10 = $-q^3$ $-q^2$ +16q – 10

π' = $-3q^2$ - 2q + 16 = 0

$$q = \frac{2 \pm \sqrt{4 - 4(-3)(16)}}{2(-3)} = \frac{2 \pm \sqrt{196}}{-6} = \frac{2 \pm 14}{-6}$$

$\Rightarrow$ q = -16/6 rejected , q = 2

10

π'' = -6q – 2 and $\pi''(2)$ = -14 < 0

$\Rightarrow$ q = 2 maximizes the profit and the maximum profit is

π(2) = -8 – 4 + 32 – 10 = 10

(b) Determine the following integrals

$$\int \sin^2 x \cos^5 xdx = \int \sin^2 x \cos^4 x \cos xdx$$

u = sinx $\Rightarrow$ du = cosx dx , the integral becomes

$$\int \sin^2 x(\cos^2 x)^2 du = \int u^2(1-\sin^2 x)^2 du$$

$$= \int u^2(1-u^2)^2 du = \int u^2(1-2u^2+u^4)du$$

$$= \int (u^2 - 2u^4 + u^6)du = \frac{u^3}{3} - 2\frac{u^5}{4} + \frac{u^7}{7} + C$$

$$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

6

$$\int \frac{e^x dx}{\sqrt{e^x+1}} \quad u = e^x + 1 \Rightarrow du = e^x dx$$

$$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = 2\sqrt{u} + C = 2\sqrt{e^x+1} + C$$ **4**

9. (a) A monopoly has fixed costs of **10** and Average variable cost function **q^2+4** .the demand equation for its product is **p+q =20**.Determine the profit function in terms of q. Determine also the production level that maximises the profit.

VC = q x AVC = $q(q^2 + 4) = q^3 + 4q$

TC = VC + FC = $q^3 + 4q + 10$

TR = pq , but p + q = 20 $\Rightarrow$ p = 20 – q

TR = (20 – q)q = $20q - q^2$

π = TR – TC = $20q - q^2 - q^3 - 4q = -q^3 - q^2 + 16q$

π' = $-3q^2 - 2q + 16 = 0$

$$q = \frac{2 \pm \sqrt{4-4(-3)(16)}}{2(-3)} = \frac{2 \pm \sqrt{196}}{-6} = \frac{2 \pm 14}{-6}$$

$\Rightarrow$ q = -16/6 rejected , q = 2 **10**

π'' = -6q – 2 and $\pi''(2)$ = -14 < 0

$\Rightarrow$ q = 2 maximizes the profit and the maximum profit is

$\pi(2)$ = -8 – 4 + 32 = 20

(b) Find the critical points of the function and specify their nature:

$$f(x) = x^4 - 8x^3 - 80x^2 + 15$$

$f`(x) = 4x^3 - 24x^2 - 160x = 0 \Rightarrow 4x(x^2-6x-40) = 0$

x = 0 , $x^2 - 6x - 40 = 0 \Rightarrow x = \frac{6 \pm \sqrt{196}}{2} = \frac{6 \pm 14}{2}$

x = 10 , x = - 4 **10**

f”(x) = $12x^2 - 48x - 160$

at x = 0 , f”(0) = -160 < 0 , maximum : (0 , 15)

at x = 10 , f”(10) = 560 > 0 , minimum : (10 , -5985)

at x = - 4 , f”(-4) = 224 > 0 , minimum : (-4 , -497)

10.(a) A firm’s marginal revenue function is **MR=11 – q** The firm’s marginal cost function is **MC = $3q^2$+ 36q – 36** where q is either the quantity sold or produced. Find the value of q which maximises the profit. Determine the maximum profit and verify that it is a maximum.

TC = $\int (3q^2 + 36q - 36)dq$ = $q^3 + 18q^2 - 36q + C$

TR = $\int (11 - q)dq$ = $11q - q^2/2$

π = TR – TC = $11q - q^2/2 - q^3 - 18q^2 + 36q - C$

π = $-q^3 - (37/2)q^2 + 47q - C \Rightarrow \pi' = -3q^2 - 37q + 47 = 0$

$\Rightarrow 3q^2 + 37q - 47 = 0$

q = $\dfrac{-37 \pm \sqrt{37^2 - 4(3)(-47)}}{2(3)} = \dfrac{-37 \pm \sqrt{1933}}{6}$

q = $\dfrac{-37 - \sqrt{1933}}{6}$ which is economically not feasible,

q = $\dfrac{-37 + \sqrt{1933}}{6}$ ($\sqrt{1933} \approx 44$)

10

$\pi'' = -6q - 37 = -2(\dfrac{-37 + \sqrt{1933}}{6}) - 37 < 0$,

q = $\dfrac{-37 + \sqrt{1933}}{6}$ maximises the profit

Another method

MR = MC , $11 - q = 3q^2 + 36q - 36 \Rightarrow 3q^2 + 37q - 47 = 0$

(b) **Show that :** $\dfrac{2x^2 - 3x + 4}{x - 1} = 2x - 1 + \dfrac{3}{x - 1}$

$$2x - 1 + \frac{3}{x-1} = \frac{(2x-1)(x-1) + 3}{x-1} = \frac{2x^2 - 3x + 4}{x-1}$$

Then find : $\int \dfrac{2x^2 - 3x + 4}{x - 1}\, dx = \int (2x - 1 + \dfrac{3}{x-1})dx$

$= x^2 - x + 3\ln|x-1| + C$

10

11. A firm faces a total cost function $TC = 20 + 5q + 5q^2$

(i) Determine the firm's average cost (AC) and marginal cost (MC) functions.

AC = $\dfrac{TC}{q} = \dfrac{20}{q} + 5 + 5q$

4

MC = TC' = 5 + 10q

(ii) Find the quantity that minimises the Average cost and the value of this minimum. Show indeed it is a minimum.

AC' = $\dfrac{-20}{q^2} + 5 = \dfrac{5q^2 - 20}{q^2}$

AC' = 0 $\Rightarrow 5q^2 - 20 = 0 \Rightarrow q^2 = 4 \Rightarrow q = -2$ (rejected) or q = 2

AC’’ = $\dfrac{40}{q^3}$ and AC’’(2) = 40/8 = 5 > 0 $\Rightarrow$ q = 2

minimizes AC and the value of the minimum is AC(2) = 10 + 5 + 10 = 25

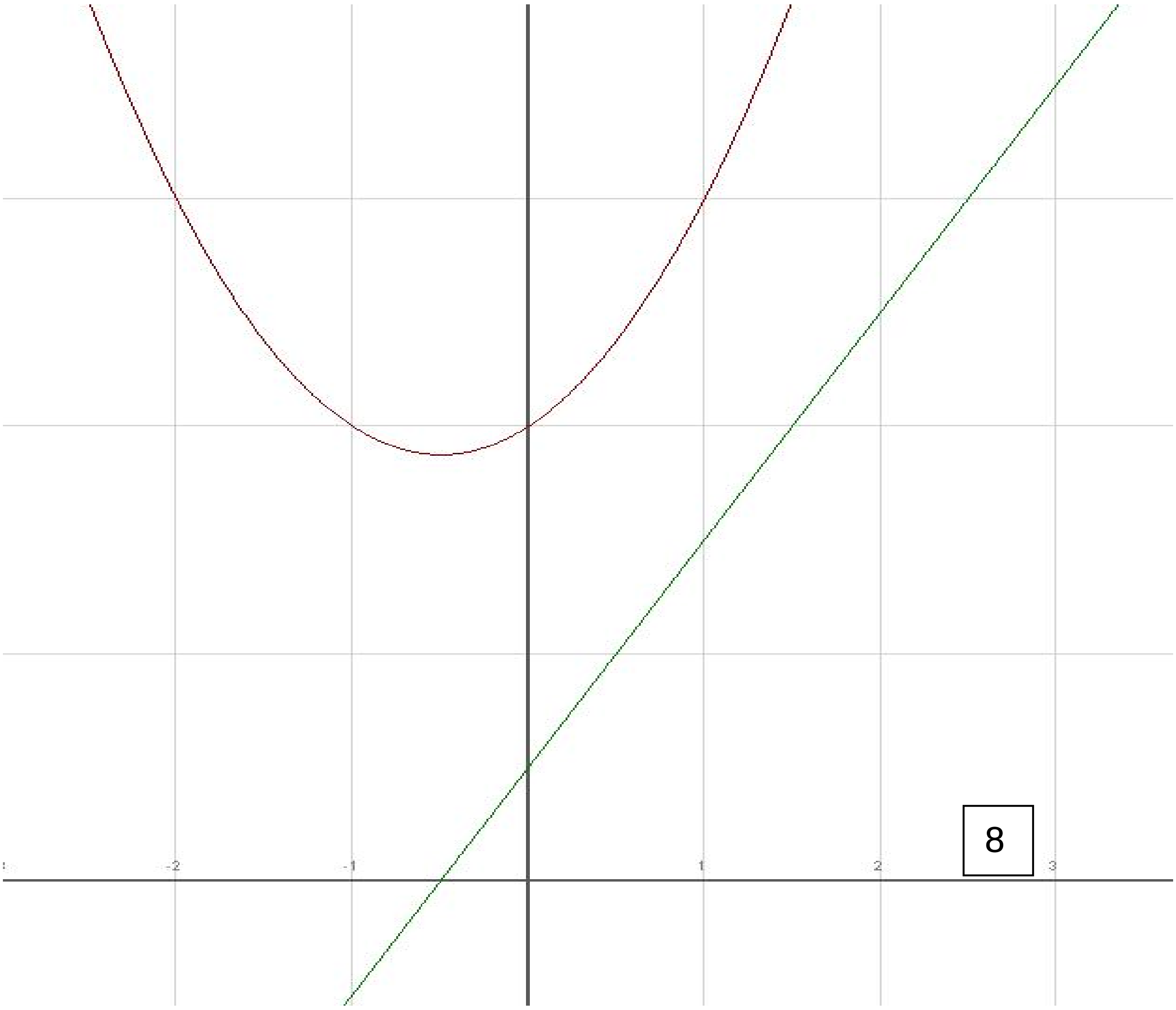
6

(iii) Verify that when q = 2 , the marginal cost MC equals the Average cost.

2

AC(2) = 25 , MC(2) = 5 + 20 = 25 $\therefore$ AC = MC at q = 2

(iv) Sketch the graphs of the total cost TC and the marginal cost MC functions on the same system of axes.



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