



Sequences & Series

Study Guide pp 120-125

Topic	Interpretation
<p>Arithmetic Sequence (Progression) $a, a+d, a+2d, a+3d, \dots, a+(n-1)d$ n^{th} term : $a_n = a + (n-1)d$ Sum of first n terms:</p> $S_n = \frac{n}{2}(a + a_n)$ $= \frac{n}{2} \{ 2a + (n-1)d \}$ <p><u>Example 3:</u> Find the sum of the first n odd positive integers: $1+3+5+7+\dots+$</p> <p>Geometric Sequence (Progression) $a, ar, ar^2, ar^3, \dots, ar^{n-1}$ n^{th} term : $a_n = ar^{n-1}$ Sum of first n terms:</p> $S_n = a \times \frac{r^n - 1}{r - 1}$ <p>Sum to infinity of a G.P. A geometric sequence is said to be infinite when $-1 < r < 1$ In this case :</p> $S_\infty = \frac{a}{1-r}$ <p><u>Example 6:</u> Find the sum to infinity of : $1 + (1/2) + (1/2)^2 + (1/2)^3 + \dots$</p>	<p><u>Example 1:</u> Fourth term = $a+3d$ 15th term = $a + 14d$ 100th term = $a + 99d$</p> <p><u>Example 2:</u> For the progression: 3,7,11,15,..... Find the 50th term. $a = 3 ; d = 7 - 3 = 4$ $a_{50} = a + 49d = 3 + 49(4) = 199$ Find the sum of the first 20 terms of the above sequence :</p> $S_{20} = (20/2)[2(3) + (20-1)(4)]$ $= (10)[6+57] = 630$ <p>It is an A.P. of first term $a=1$ and common difference $d = 2$ $S = (n/2)[2(1) + (n-1)(2)]$ $= (n/2)[2n] = n^2$</p> <p><u>Example 4:</u> 7th term = ar^6 40th term = ar^{39}</p> <p><u>Example 5:</u> For the sequence 3,6,12,..... Find the 10th term. $a = 3 , r = 6/3 = 2$ $a_{10} = ar^9 = 3(2)^9$ Find the sum of the first 15 terms:</p> $S_n = a \times \frac{r^n - 1}{r - 1} ; S_{15} = 3 \times \frac{2^{15} - 1}{2 - 1}$ $S_{15} = 3(2^{15} - 1)$ <p>$\Rightarrow a = 1 , r = 1/2 < 1$ $S = a/(1-r) = 1/(1-1/2) = 2$</p>

Compound Interest

If **P** amount is deposited in an account where the interest is paid annually at a *rate r*, then after **t** years, we have a balance of $P(1+r)^t$

Rate Distributed

If the year is divided over **m** equal periods, the rate r/m over each period and the balance becomes $P(1+r/m)^m$

Compound interest and Exponential :

Continuous compounding :

when **m** tends to infinity :

As $m \rightarrow \infty$, $(1+r/m)^m = e^r$

The balance after one year:

$$P(1+r/m)^m = Pe^r$$

For another year : $(Pe^r)(e^r) = Pe^{2r}$

After **t** years: Pe^{rt}

Solution of Difference Equations

(1) First order :

have difference equation

$$y_k = ay_{k-1} + b \text{ with initial value } y_0$$

$$k=1 : y_1 = ay_0 + b$$

$$k=2:$$

$$y_2 = ay_1 + b = a[ay_0 + b] + b$$

$$y_2 = a^2 y_0 + b(1+a)$$

$$k=3 : y_3 = ay_2 + b$$

$$= a[a^2 y_0 + b(1+a)] + b$$

$$y_3 = a^3 y_0 + b(1+a+a^2)$$

$$k=n : y_n = ay_{n-1} + b$$

Examples: $y_k = 2y_{k-1} - 5 ; y_0 = y_0$

$$y_n = a^n y_0 + b(1 - a^n / 1 - a)$$

$$a = 2 ; b = -5$$

$$y_n = 2^n y_0 - 5(1 - 2^n / 1 - 2)$$

$$y_n = 2^n y_0 + 5(1 - 2^n) =$$

$$= 2^n y_0 + 5 - 5 \times 2^n = 5 + 2^n (y_0 - 5)$$

Example 7:

Suppose that \$1000 is invested in an account that pays interest at a fixed rate of 7% paid annually. How much is there in the account after 4 years?

$$P(1+r)^t = 1000(1+0.07)^4 = \$1310.80$$

Example 8:

Suppose that \$100 is invested in an account that pays interest at a fixed rate of 8% paid annually. How much is there in the account by the end of the year?

$$P(1+r)^t = 100(1+0.08)^1 = \$108$$

*Suppose that the interest is added **twice** yearly. the balance : $r = 2$

$$P(1+r/m)^m = 100(1+0.08/2)^2 = 100(1+0.04)^2 = \$108.16$$

*Suppose that the interest is paid

quarterly : $r = 4$

$$P(1+r/m)^m = 100(1+0.08/4)^4 = 100(1+0.02)^4 = \$108.24$$

Example 9:

Suppose \$ 5000 is invested at an annual rate of 4% **compounded continuously** for 5 years. Find the compound amount.

$$P = 5000, r = 0.04, t = 5$$

$$Pe^{rt} = 5000e^{(0.04)(5)} = 5000e^{0.2} = \$6107.01$$

$$y_n = a[a^{n-1}y_0 + b(1+a+a^2+\dots+a^{n-2})] + b$$

$$y_n = a^n y_0 + b(1+a+a^2+\dots+a^{n-1})$$

$1+a+a^2+\dots+a^{n-1}$ is a GP of first term 1, number of terms **n** and common ratio $r = a$:

$$1+a+a^2+\dots+a^{n-1} =$$

$$a_1 \times (1 - r^n / 1 - r) = 1 \times (1 - a^n / 1 - a),$$

However if $a = 1$ then $1+a+a^2+\dots+a^{n-1} = n$

$$y_n = y_0 + bn \text{ if } a = 1$$

$$y_n = a^n y_0 + b(1 - a^n / 1 - a) \text{ if } a \neq 1$$