International Institute
for Technology and Management

Unit 05a : Mathematics 1 **Handout #16**

Partial Derivatives II

Study Guide pp 83-86 ;Anthony & Biggs pp 148-149

Topic	Interpretation
Chain Rule : Let f be a function in x and y where x and y are functions of t.Let F(x(t),y(t)) ,then dF/dt is called the total derivative of F with respect to t is : $\frac{dF}{dt}=\frac{\partial f}{\partial x}\times\frac{dx}{dt}+\frac{\partial f}{\partial y}\times\frac{dy}{dt}$ <u>Example1:</u> Let $f(x,y) = x^2y$ where $x = t + 3$ and $y = 2t$ if $F(t) = f(x(t),y(t))$ find the derivative dF/dt	$f(x,y) = x^2y$ $\frac{\partial f}{\partial x} = 2xy$; $\frac{dx}{dt} = 1$ $\frac{\partial f}{\partial y} = x^2$; $\frac{dy}{dt} = 2$ $\frac{dF}{dt}=\frac{\partial f}{\partial x}\times\frac{dx}{dt}+\frac{\partial f}{\partial y}\times\frac{dy}{dt}$ $\frac{dF}{dt} = (2xy)(1) + (x^2)(2)$ $= 2(t+3)(2t) + 2(t+3)^2$ $= 6t^2 + 24t + 18$
Implicit Partial Differentiation: Let $g(x,y) = c$ Then $\frac{dy}{dx}=-\frac{\partial g/\partial x}{\partial g/\partial y}$ <u>Example2:</u> Consider the function $e^{x^2y} = 7$ Find $\frac{dy}{dx}$	Let $g(x,y) = e^{x^2y}$ $\frac{\partial g}{\partial x} = 2xy\,e^{x^2y}$ $\frac{\partial g}{\partial y} = x^2\,e^{x^2y}$ $\frac{dy}{dx}=-\frac{\partial g/\partial x}{\partial g/\partial y} = -\frac{2xye^{x^2y}}{x^2e^{x^2y}} = -\frac{2y}{x}$
Optimization Let f(x,y) be a function of two variables. The Critical point is obtained by Solving $f_1=0;f_2 = 0$	<u>Example3:</u> Find the critical point(s) of $f(x,y) = y^3 + 3xy - x^3$ $f_1 = 3y-3x^2 = 0 \Rightarrow y = x^2$ $f_2 = 3y^2+ 3x = 0 \Rightarrow y^2 + x = 0$ $\Rightarrow (x^2)^2 + x = 0 \Rightarrow x^4 + x = 0$

Nature of Critical points :

1. Maximum /Minimum:

$(f_{11})(f_{22}) - f_{12}^2 > 0$

If $f_{11} > 0 \Rightarrow$ Minimum

If $f_{11} < 0 \Rightarrow$ Maximum

2. Saddle point :

$(f_{11})(f_{22}) - f_{12}^2 < 0$

Example4:

Find and classify the critical points of the following functions :

a. $f(x,y)=x^2 + y^2$

b. $f(x,y)=3-6x+x^2+2xy+2y - y^2$

c. $f(x,y)=Ln(1+x^2 + y^2)$

d. $f(x,y) = e^{x^2+y^2}$

$f_1= 2xe^{x^2+y^2} = 0 \Rightarrow x = 0$; $f_2= 2ye^{x^2+y^2} = 0 \Rightarrow y = 0$;

(0,0) is the critical point.

$f_{11} =2e^{x^2+y^2} +(2x)(2x)e^{x^2+y^2}$

$=(4x^2 +2)e^{x^2+y^2}$

[Using $u = 2x$; $v = e^{x^2+y^2}$ then $u'v + v'u$]

With $x = 0, y = 0$; $f_{11}=2e^0 = 2$

$f_{22} =2e^{x^2+y^2} +(2y)(2y)e^{x^2+y^2}$

$= (4y^2+2)\ e^{x^2+y^2}$

with $x = 0, y = 0$; $f_{22} =2e^0 = 2$

$f_{12} = (2x)(2y)e^{x^2+y^2}$;

with $x = 0$, $y= 0$; $f_{12} = 0$

$(f_{11})(f_{22}) - f_{12}^2 = 4 > 0$;Since $f_{11} = 2 > 0$,(0,0) minimizes f .

$\Rightarrow x(x^3 +1) = 0$

$\Rightarrow x(x+1)(x^2 -x +1) =0 \Rightarrow x =0$ or $x=- 1$

$x = 0 \Rightarrow y = x^2 = 0 \Rightarrow (0 , 0)$

$x = -1 \Rightarrow y = x^2 = 1 \Rightarrow (-1,1)$

Example4:

a. $f(x,y)=x^2 + y^2$

$f_1= 2x = 0 \Rightarrow x = 0$

$f_2= 2y = 0 \Rightarrow y = 0$

(0 , 0) is the critical point .

$f_{11} = 2$, $f_{22} = 2$; $f_{12} = 0$

$\Rightarrow (f_{11})(f_{22}) - f_{12}^2 = 4 > 0$

Since $f_{11} = 2 > 0 \Rightarrow (1,1)$ minimizes f .

b. $f(x,y)=3 - 6x+x^2 +2xy +2y- y^2$

$f_1= - 6 +2x + 2y = 0$

$f_2= 2x + 2 - 2y = 0$

Solving simultaneously, $x = 1$, $y = 2$

$\Rightarrow$ (1,2) is the critical point.

$f_{11} = 2$, $f_{22} = -2$; $f_{12} = 2$

$\Rightarrow (f_{11})(f_{22}) - f_{12}^2 = -8 < 0$

$\Rightarrow$ (1,2) is a saddle point .

c. $f(x,y)=Ln(1+ x^2 + y^2)$

$f_1=\frac{2x}{1+x^2+y^2}=0 \Rightarrow x = 0$

$f_2=\frac{2y}{1+x^2+y^2}=0 \Rightarrow y = 0$

(0,0) is the critical point .

$f_{11} = \frac{2(1+x^2+y^2)-2x(2x)}{(1+x^2+y^2)^2}$

with $(x,y) =(0,0) \Rightarrow f_{11} = 2$

$f_{22} = \frac{2(1+x^2+y^2)-2y(2y)}{(1+x^2+y^2)^2}$

with $(x,y) =(0,0) \Rightarrow f_{22} = 2$

$f_{12} = \frac{(0)(1+x^2+y^2)-2x(2y)}{(1+x^2+y^2)^2}$

with $(x,y) =(0,0) \Rightarrow f_{12} = 0$

$\Rightarrow (f_{11})(f_{22}) - f_{12}^2 = 4 > 0$

Since $f_{11} = 2 > 0 \Rightarrow (0,0)$ minimizes f .