SECTION A

Answer all **SEVEN** questions from this section (60 marks in total)

1. Sketch the graphs of the functions f and g where

$$f(x) = -x^2 + 3x + 10 \qquad \text{and} \qquad g(x) = 2x + 4,$$

and determine their points of intersection.

(1) It has ∩ shape since it has negative x^2 term

(2) Intercepts: x-intercepts : y = 0 , $-x^2 + 3x + 10 = 0$, $x = \dfrac{-3 \pm \sqrt{49}}{-2}$

x-intercpts: x = 5 or x = -2 $\Rightarrow$ (5,0) ; (-2,0)

y-intercepts: x = 0 $\Rightarrow$ y = 10 : (0 , 10)

(3) The maximum : y' = -2x +3 = 0 $\Rightarrow$ x = 3/2 $\Rightarrow$ y = 49/4 $\Rightarrow$ V(3/2 , 49/4)

OR, $x = \dfrac{-b}{2a} = \dfrac{-3}{-2} = \dfrac{3}{2}$ $\Rightarrow$ y = 49/4 $\Rightarrow$ V(3/2 , 49/4)

g(x) = 2x + 4

x	-2	-1	0	1
y	0	2	4	6

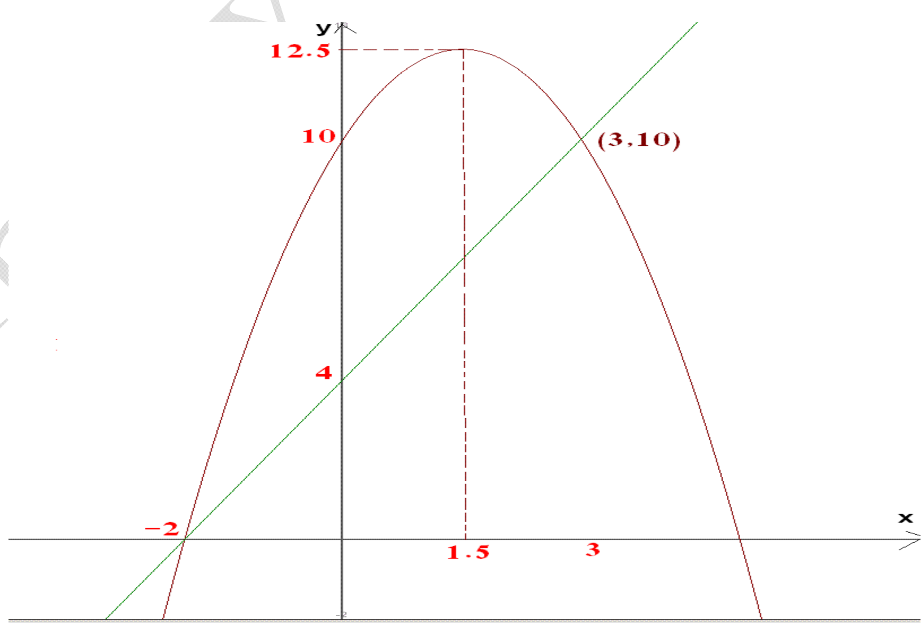
Points of intersection: f(x) = g(x)

$-x^2 + 3x + 10 = 2x + 4 \Rightarrow -x^2 + x + 6 = 0 \Rightarrow x = -2$ or $x = 3$

x = -2 $\Rightarrow$ y = 2x+4 = 2(-2) + 4 = 0 , 1st point (-2,0)

x = 3 $\Rightarrow$ y = 2x+4 = 2(3) + 4 = 10 , 2nd point (3,10)

Graphs:



2. Find and classify the stationary points of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 9$.

$f'(x) = 12x^3 - 12x^2 - 24x = 0 \Rightarrow 12x(x^2 - x - 2) = 0$
$\Rightarrow 12x = 0 \Rightarrow x = 0 \Rightarrow y = 3(0)^4 - 4(0)^3 - 12(0)^2 + 9 = 9 \Rightarrow$ 1st point (0 , 9)
or $x^2 - x - 2 = 0 \Rightarrow x = -1$ or $x = 2$
$x = -1 \Rightarrow y = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 9 = 4 \Rightarrow$ 2nd point (-1 , 4)
$x = 2 \Rightarrow y = 3(2)^4 - 4(2)^3 - 12(2)^2 + 9 = -23 \Rightarrow$ 3rd point (2 , -23)
<u>Nature:</u>

$f''(x) = 36x^2 - 24x - 24$
at $x = 0 \Rightarrow f''(0) = -24 < 0 \Rightarrow (0, 9)$ maximizes f(x)
at $x = -1 \Rightarrow f''(-1) = 36 > 0 \Rightarrow (-1, 4)$ minimizes f(x)
at $x = 2 \Rightarrow f''(2) = 72 > 0 \Rightarrow (2, -23)$ minimizes f(x)

3. **Use a matrix method** to find the numbers x, y and z that satisfy the equations $x + y + z = 4$, $2x - y - z = 2$ and $-x + y + 2z = 3$.

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & -1 & -1 & 2 \\ -1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{2I_1 - I_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & 6 \\ -1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{I_1 + I_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & 6 \\ 0 & 2 & 3 & 7 \end{bmatrix} \xrightarrow{2I_2 - 3I_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

$\therefore -3z = -9 \Rightarrow z = 3$
$3y + 3z = 6 \Rightarrow 3y + 3(3) = 6 \Rightarrow 3y = 6 - 9 = -3 \Rightarrow y = -1$
$x + y + z = 4 \Rightarrow x - 1 + 3 = 4 \Rightarrow x = 2$
$(x, y, z) = (2, -1, 3)$

4. Suppose that the constant $a \neq -2, 0, 2$. Show that the only critical point of the function

$$f(x, y) = x^2 + axy + y^2,$$

is $(0, 0)$. Determine, for each possible value of a, the nature of this critical point.

$f_x = 2x + ay = 0$ and $f_y = ax + 2y = 0 \Rightarrow y = -ax/2$ substitute this in f_x
$\Rightarrow 2x + a(-ax/2) = 0 \Rightarrow 4x - a^2x = 0 \Rightarrow (4 - a^2)x = 0$
Since $a \neq -2, 0, 2 \Rightarrow 4 - a^2 \neq 0 \Rightarrow x = 0 / 4 - a^2 = 0$
$x = 0 \Rightarrow y = -ax/2 = 0 \Rightarrow (0, 0)$ is the critical point.
<u>Nature:</u> $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = a \Rightarrow (f_{xx})(f_{yy}) - (f_{xy})^2 = 4 - a^2 = (2 - a)(2 + a)$

-	-2	+	2	-

for $a < -2$ or $a > 2$: $(f_{xx})(f_{yy}) - (f_{xy})^2 = 4 - a^2 < 0 \Rightarrow$ Saddle

For $-2 < a < +2$: $(f_{xx})(f_{yy}) - (f_{xy})^2 = 4 - a^2 > 0$ and $f_{xx} = 2 > 0 \Rightarrow$ Minimum.

5. Determine the following integrals.

(a) $\int x^n \ln(x)\,dx$, where $n \neq -1$.

(b) $\int \frac{\cos x}{1-\sin^2 x}\,dx$.

(a) $\int x^n \ln x\, dx$, by parts

$u = \ln x \Rightarrow du = \frac{dx}{x}$

$dv = x^n\,dx \Rightarrow v = \frac{x^{n+1}}{n+1}$

$$\int u\,dv = uv - \int v\,du \Rightarrow \int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\times\frac{dx}{x}$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^{n+1}\times x^{-1}dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\times\frac{x^{n+1}}{n+1} + c = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

(b) $\int \frac{\cos x}{1-\sin^2 x}dx$ Let $u = \sin x \Rightarrow du = \cos x\,dx$

$= \int \frac{du}{1-u^2}$, by partial fractions: $\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{a}{1-u} + \frac{b}{1+u}$

$\Rightarrow 1 = a(1+u) + b(1-u)$

Choose $u = -1 \Rightarrow 1 = a(0) + 2b \Rightarrow b = \frac{1}{2}$

Choose $u = 1 \Rightarrow 1 = 2a + b(0) \Rightarrow a = \frac{1}{2}$

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int\frac{du}{1-u} + \frac{1}{2}\int\frac{du}{1+u} = -1/2\ln|1-u| + 1/2\ln|1+u| + c$$

(using $\int \frac{du}{au+b} = \frac{1}{a}\ln|au+b| + c$)

$$= \tfrac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + c = \tfrac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c$$

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6. Find the minimum value of $\frac{1}{x} + \frac{1}{y}$ for positive x and y satisfying $x^2 + y^2 = 8$.

Lagrange : $L = f - \lambda g = \frac{1}{x} + \frac{1}{y} - \lambda(x^2 + y^2 - 8)$

$$L_x = \frac{-1}{x^2} - 2\lambda x = 0 \Rightarrow \lambda = \frac{-1}{x^3}$$

$$L_y = \frac{-1}{y^2} - 2\lambda y = 0 \Rightarrow \lambda = \frac{-1}{y^3}$$

$\lambda = \lambda \Rightarrow \frac{-1}{x^3} = \frac{-1}{y^3} \Rightarrow x^3 = y^3 \Rightarrow x = y$ substitute for y in g :

$x^2 + y^2 = 8 \Rightarrow x^2 + x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

but $x > 0$, $y > 0$ (given) $\Rightarrow x = y = 2$

Value of the minimum: $\frac{1}{x} + \frac{1}{y} = \frac{1}{2} + \frac{1}{2} = 1$

7. Given that $V(x, y) = xe^{x+ay}$, find the partial derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$.

Suppose that V satisfies

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} = \left(3 + \frac{1}{x}\right) V,$$

find the value of the constant a.

$\frac{\partial V}{\partial x} = (1)(e^{x+ay}) + (x)(e^{x+ay}) = (1 + x)\, e^{x+ay}$, $\frac{\partial V}{\partial y} = ax\, e^{x+ay}$

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} = \left(3 + \frac{1}{x}\right)V \Rightarrow (1 + x)\, e^{x+ay} + ax\, e^{x+ay} = \left(3 + \frac{1}{x}\right)x\, e^{x+ay}$$

$$\Rightarrow (1 + x + ax)\, e^{x+ay} = \left(3 + \frac{1}{x}\right)x\, e^{x+ay} \Rightarrow 1 + x + ax = \left(3 + \frac{1}{x}\right)x$$

$\Rightarrow (a + 1)x + 1 = 3x + 1$, comparing the coefficient of x :

$a + 1 = 3 \Rightarrow a = 2$

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SECTION B

Answer **TWO** questions from this section (20 marks each)

8. (a) For $0 \le q \le 5$, a firm has a marginal cost function given by

$$MC(q) = 3q^2 + 36q - 36,$$

and its fixed costs are 6. If its revenue function is

$$2q^3 + 6q^2 + 9q,$$

find the profit function of the firm and determine the value of q that gives the maximum profit.

(b) Find the positive number y which is such that

$$\int_1^y \left(1 + \frac{2}{x^2}\right) dx = 2.$$

(a) TC = $\int MC dq = \int (3q^2 + 36q - 36)dq$ = q3 + 18q2 - 36q + C

FC = TC(0) = 40 $\Rightarrow$ 0 + 0 + 0 + C = 6 , C = 6

TC = q3 + 18q2 - 36q + 6

π = TR – TC = 2q3 + 6q2 +9q - q3 - 18q2 + 36q - 6 = q3 - 12q2 +45q – 6

π' = 3q2 - 24q + 45 = 0 $\Rightarrow$ q2 – 8q + 15 = 0

$$q = \frac{8 \pm \sqrt{64 - 4(1)(15)}}{2} = \frac{8 \pm \sqrt{4}}{2} = \frac{8 \pm 2}{2} \Rightarrow q = 5 \ , q = 3$$

π'' = 6q – 24 , π'' (5) = 6(5) – 24 = 6 > 0 $\Rightarrow$ q = 5 minimises the profit.

π'' = 6q – 24 , π'' (3) = 6(3) – 24 = - 6 < 0 $\Rightarrow$ q = 3 maximises the profit.

(b) $\int_1^y (1 + 2x^{-2})dx = 2 \Rightarrow \left(x + \frac{2x^{-2+1}}{-2+1}\right)_1^y = 2 \Rightarrow \left(x - \frac{2}{x}\right)_1^y = 2$

$\Rightarrow \left(y - \frac{2}{y}\right) - \left(1 - \frac{2}{1}\right) = 2 \Rightarrow y - \frac{2}{y} + 1 = 2 \Rightarrow y^2 - y - 2 = 0$

$\Rightarrow$ y = 2 or y = -1 which is rejected since y > 0 $\Rightarrow$ y = 2

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9. (a) Find the critical points of the function

$$f(x,y) = 3x^3 + 9x^2 - 72x + 2y^3 - 12y^2 - 126y + 19,$$

and determine their natures.

(a) $f(x,y) = 3x^3 + 9x^2 - 72x + 2y^3 - 12y^2 - 126y + 19$

$f_x = 6x^2 + 18x - 72 = 0 \Rightarrow 2x^2 + 6x - 8 = 0 \Rightarrow x = 1$ or $x = -4$

$f_y = 6y^2 - 24y - 126 = 0 \Rightarrow y^2 - 4y - 21 = 0 \Rightarrow y = -3$ or $y = 7$

The critical points : (1 , -3) , (1 , 7) , (- 4 , - 3) and (- 4 , 7)

$f_{xx} = 12x + 18$, $f_{yy} = 12y - 24$, $f_{xy} = 0$

at (1, -3) :

$\Rightarrow (f_{xx})(f_{yy}) - (f_{xy})^2 = (30)(-60) - 0 < 0 \Rightarrow$ (1, -3) is a Saddle point.

at (1, 7) :

$\Rightarrow (f_{xx})(f_{yy}) - (f_{xy})^2 = (30)(60) - 0 > 0$ and $f_{xx} = 30 > 0$

$\Rightarrow$ (1, 7) is a minimum point.

at (-4, -3) :

$\Rightarrow (f_{xx})(f_{yy}) - (f_{xy})^2 = (-30)(-60) - 0 > 0$ and $f_{xx} = -30 < 0$

$\Rightarrow$ (-4, -3) is a maximum point.

at (- 4, 7) :

$\Rightarrow (f_{xx})(f_{yy}) - (f_{xy})^2 = (-30)(60) - 0 < 0 \Rightarrow$ (-4, 7) is a Saddle point.

(b) The numbers x, y, z and a are related by the equations

$$x + 2y - z = 3$$
$$-x - 3y + z = -5$$
$$2x + y - z = a$$

Using a matrix method, find expressions for x, y, z in terms of a.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ -1 & -3 & 1 & -5 \\ 2 & 1 & -1 & a \end{bmatrix} \xrightarrow{I_1+I_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -2 \\ 2 & 1 & -1 & a \end{bmatrix} \xrightarrow{2I_1-I_3}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 3 & -1 & 6-a \end{bmatrix} \xrightarrow{3I_2+I_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -a \end{bmatrix}$$

$\therefore -z = -a \Rightarrow z = a$

$-y = -2 \Rightarrow y = 2$

$x + 2y - z = 3 \Rightarrow x + 4 - a = 3 \Rightarrow x = a - 1$

$(x , y , z) = (a - 1 , 2 , a)$

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10. (a) A village initially contains 100 women. Each year, of the women who were in the village at the start of the year, 20% give birth to a daughter and 10% of the women die. Additionally, each year, 5 new women come to the village from the surrounding countryside. How many women will there be in the village after t years?

20% borned and 10% died $\Rightarrow$ 10 % survive (are added)

At the start of year 1 : $W_1 = 100(1.1)^1 + 5$

At the start of year2 : $W_2 = W_1(1.1) + 5 = [100(1.1)^1 + 5](1.1) + 5$
$= 100(1.1)^2 + 5(1.1) + 5$

At the start of year3 : $W_3 = W_2(1.1) + 5$
$= [\ 100(1.1)^2 + 5(1.1) + 5](1.1) + 5$
$= 100(1.1)^3 + 5(1.1)^2 + 5(1.1) + 5$

At the start of year t : $W_t = 100(1.1)^t + 5(1.1)^{t-1} + 5(1.1)^{t-2} + \ldots\ldots\ldots + 5$

$W_t = 100(1.1)^t + 5[1 + (1.1)^1 + (1.1)^2 + \ldots\ldots + (1.1)^{t-2} + (1.1)^{t-1}]$

$$W_t = 100(1.1)^t + 5[1\times\frac{(1.1)^t - 1}{1.1-1}] = 100(1.1)^t + 5[1\times\frac{(1.1)^t - 1}{0.1}]$$

$W_t = 100(1.1)^t + 50\,[(1.1)^t - 1]$.

(b) A firm is the only producer of two goods, X and Y. The demand functions for X and Y are, respectively, given by

$$x = 8(p_Y - p_X) \qquad \text{and} \qquad y = 4(9 + 2p_X - 4p_Y),$$

where p_X and p_Y are the corresponding prices. If it costs \$1 to produce one unit of X and \$1.50 to produce one unit of Y, find an expression for the profit function of the firm in terms of x and y. Hence determine the quantities x and y that maximise the profit.

The cost function = (1)(x) + (1.5)(y) = x + 1.5y

x = 8(Py – Px) , y = 4(9 + 2Px - 4 Py)

Rearrange as simultaneous. Equations in Px and Py

-8Px + 8Py = x ,

8Px -16Py = y – 36 . Adding the equations:

$$-8Py = x + y - 36 \Rightarrow Py = \frac{-x - y + 36}{8} \text{ and } Px = \frac{-2x - y + 36}{8}$$

$$\text{Total Revenue} = xPx + yPy = x\left(\frac{-2x - y + 36}{8}\right) + y\left(\frac{-x - y + 36}{8}\right)$$

$TR = -x^2/4 - xy/8 + 9/2\ x - xy/8 - y^2/8 + 9/2\ y = -x^2/4 - xy/4 + 9/2\ x - y^2/8 + 9/2\ y$

$\pi = TR - TC = -x^2/4 - xy/4 + 9/2\ x - y^2/8 + 9/2\ y - x - 1.5y$

$\pi_x = -\frac{1}{2}x - \frac{1}{4}y + 9/2 - 1 = 0 \Rightarrow -2x - y + 14 = 0$

$\pi_y = -\frac{1}{4}x - \frac{1}{4}y + 9/2 - 1.5 = 0 \Rightarrow -x - y + 12 = 0$

Solving simultaneously for x and y : x = 2 , y = 10

$\pi_{xx} = -\frac{1}{2}$, $\pi_{yy} = -\frac{1}{4}$, $\pi_{xy} = -\frac{1}{4}$

$\Rightarrow (\pi_{xx})(\pi_{yy}) - (\pi_{xy})^2 = (-\frac{1}{2})(-\frac{1}{4}) - (-\frac{1}{4})^2 = 1/8 - 1/16 = 1/16 > 0$

and since $\pi_{xx} = -\frac{1}{2} < 0 \Rightarrow$ (2 , 10) maximizes the profit.

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11. Given an amount of capital, k, and an amount of labour, l, a firm produces a quantity $q(k,l) = k^\alpha l^\alpha$ where α is a constant such that $0 < \alpha < 1/2$. Each unit of capital costs v and each unit of labour costs w.

(a) Show that the minimum amount the firm can spend on capital and labour, if it is to manufacture an amount Q, is given by

$$2\sqrt{vw}Q^{1/(2\alpha)}.$$

The product manufactured by the firm sells at a fixed price of p per unit and the raw materials required to produce each unit cost an amount r where $r < p$.

(b) Assuming that the firm is acting so as to minimise its capital and labour costs, use the result above to find an expression for the profit made by the firm if it sells an amount Q.

(c) Find the amount, Q, that will maximise the firm's profit. (Make sure that the assertion that the profit is maximised is justified.)

(a) The problem is : minimise $kv + lw$ subject to $(kl)^\alpha = Q$

$$L = f - \lambda g = kv + lw - \lambda(k^\alpha l^\alpha - Q)$$

$$\frac{\partial L}{\partial k} = v - \lambda \alpha k^{\alpha-1} l^\alpha = 0 \Rightarrow \lambda = \frac{v}{\alpha k^{\alpha-1} l^\alpha} = \frac{v}{\alpha} k^{-\alpha+1} l^{-\alpha}$$

$$\frac{\partial L}{\partial l} = w - \lambda \alpha k^\alpha l^{\alpha-1} = 0 \Rightarrow \lambda = \frac{w}{\alpha k^\alpha l^{\alpha-1}} = \frac{w}{\alpha} k^{-\alpha} l^{-\alpha+1}$$

$$\lambda = \lambda \Rightarrow \frac{v}{\alpha} k^{-\alpha+1} l^{-\alpha} = \frac{w}{\alpha} k^{-\alpha} l^{-\alpha+1} \Rightarrow v\, k^{-\alpha+1} l^{-\alpha} = w\, k^{-\alpha} l^{-\alpha+1}$$

$$\frac{k^{-\alpha+1} l^{-\alpha}}{k^{-\alpha} l^{-\alpha+1}} = \frac{w}{v} \Rightarrow k^{-\alpha+1} l^{-\alpha}\ k^\alpha l^{\alpha-1} = \frac{w}{v} \Rightarrow kl^{-1} = \frac{w}{v} \Rightarrow \frac{k}{l} = \frac{w}{v} \Rightarrow$$

$k = l\dfrac{w}{v}$, Substitute this in $(kl)^\alpha = Q \Rightarrow (\dfrac{w}{v} l^2)^\alpha = Q$

$$\Rightarrow w^\alpha v^{-\alpha} l^{2\alpha} = Q \Rightarrow l^{2\alpha} = \frac{Q}{w^\alpha v^{-\alpha}} = Q v^\alpha w^{-\alpha}$$

$$\Rightarrow l = (Qv^\alpha w^{-\alpha})^{1/2\alpha} = Q^{1/2\alpha} v^{1/2} w^{-1/2};$$

$$\Rightarrow k = l\frac{w}{v} = lwv^{-1} = Q^{1/2\alpha} v^{1/2} w^{-1/2} v^{-1} w = Q^{1/2\alpha} v^{-1/2} w^{1/2}$$

The minimum cost : $kv + lw$

$$= (Q^{1/2\alpha} v^{-1/2} w^{1/2})v + (Q^{1/2\alpha} v^{1/2} w^{-1/2})w$$

$$= Q^{1/2\alpha}(v^{1/2}w^{1/2} + v^{1/2}w^{1/2}) = Q^{1/2\alpha}(\sqrt{vw} + \sqrt{vw}) = 2\sqrt{vw}Q^{1/2\alpha}$$

(b) Each unit costs r , Q units cost rQ ,the cost is minimum :

TC = rQ + $2\sqrt{vw}Q^{1/2\alpha}$

TR = pQ

Profit = π = TR – TC = pQ – rQ - $2\sqrt{vw}Q^{1/2\alpha}$

(c) $$\frac{d\pi}{dQ} = p - r - \frac{1}{\alpha}\sqrt{vw}Q^{\frac{1-2\alpha}{2\alpha}} = 0 \Rightarrow p - r = \frac{1}{\alpha}\sqrt{vw}Q^{\frac{1-2\alpha}{2\alpha}}$$

$$\Rightarrow \frac{(p-r)\alpha}{\sqrt{vw}} = Q^{\frac{1-2\alpha}{2\alpha}} \Rightarrow Q = \left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}}$$

$$\frac{d^2\pi}{dQ^2} = -\frac{1}{\alpha}\left(\frac{1-2\alpha}{2\alpha}\right)\sqrt{vw}Q^{\frac{1-4\alpha}{2\alpha}} \quad , \quad \text{At } Q = \left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}} :$$

$$\frac{d^2\pi}{dQ^2} = -\frac{1}{\alpha}\left(\frac{1-2\alpha}{2\alpha}\right)\sqrt{vw}\left[\left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}}\right]^{\frac{1-4\alpha}{2\alpha}}$$

$$\frac{d^2\pi}{dQ^2} = -\frac{1}{\alpha}\left(\frac{1-2\alpha}{2\alpha}\right)\sqrt{vw}\left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}\times\frac{1-4\alpha}{2\alpha}}$$

$$\frac{d^2\pi}{dQ^2} = -\frac{1}{\alpha}\left(\frac{1-2\alpha}{2\alpha}\right)\sqrt{vw}\left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{1-4\alpha}{1-2\alpha}}$$

$$0 \prec \alpha \prec 1/2 \Rightarrow 0 \prec 2\alpha \prec 1 \Rightarrow -1 \prec -2\alpha \prec 0 \Rightarrow 0 \prec 1-2\alpha \prec 1 \Rightarrow 1-2\alpha \succ 0$$

$$0 \prec \alpha \prec 1/2 \Rightarrow \alpha \succ 0 \Rightarrow \frac{1-2\alpha}{2\alpha} > 0 \text{ and } \frac{-1}{\alpha} < 0 \Rightarrow \frac{-1}{\alpha}\left(\frac{1-2\alpha}{2\alpha}\right) < 0$$

Moreover , r < p $\Rightarrow$ p – r > 0 , α > 0 , $\sqrt{vw} \succ 0 \Rightarrow \frac{(p-r)\alpha}{\sqrt{vw}} \succ 0$

$$\Rightarrow \left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{1-4\alpha}{1-2\alpha}} \succ 0 \Rightarrow \frac{d^2\pi}{dQ^2} = -\frac{1}{\alpha}\left(\frac{1-2\alpha}{2\alpha}\right)\sqrt{vw}\left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{1-4\alpha}{1-2\alpha}} < 0$$

Hence $Q = \left(\frac{(p-r)\alpha}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}}$ maximizes the profit.

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