SECTION A

Answer all seven questions from this section (60 marks in total).

1. Suppose that a > 0. Find the maximum value of the function
   $f(x) = \ln x - ax$
   for x > 0. Hence show that, for all x > 0, $e^{ax} \le aex$

$f'(x) = \frac{1}{x} - a = 0 \Rightarrow \frac{1}{x} = a \Rightarrow ax = 1 \Rightarrow x = \frac{1}{a}$

$f''(x) = \frac{-1}{x^2} \prec 0$ for every $x \neq 0$ $\Rightarrow x = \frac{1}{a}$ maximizes f(x)

The value of the maximum : $f(\frac{1}{a}) = \ln(\frac{1}{a}) - a(\frac{1}{a})$

With ln(1/a) = ln1 – ln a = 0 – lna = - lna, $= -\ln a - 1$

Since - lna – 1 is the maximum value , f(x) ≤ -lna -1 (x > 0)

$\Rightarrow \ln x - ax \le -\ln a - 1 \Rightarrow \ln x + \ln a \le ax - 1$

$\Rightarrow \ln(ax) \le ax - 1 \Rightarrow ax \le e^{ax-1}$ [remember : ln x = a , x = e^a]

$\Rightarrow ax \le e^{ax} \times e^{-1} \Rightarrow \frac{ax}{e^{-1}} \le e^{ax} \Rightarrow aex \le e^{ax} \Rightarrow e^{ax} \ge aex$

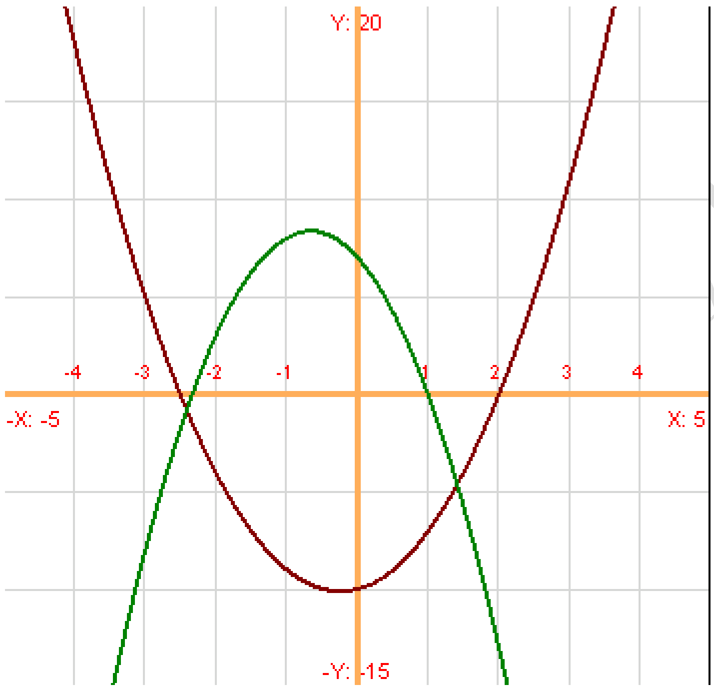
2. The functions f(x) and g(x) are
   $f(x) = 2x^2 + x - 10$, $g(x) = 7 - 3x^2 - 4x$.
   Sketch the graphs of f and g, and determine the x-coordinates of their points of intersection.
   $f(x) = 2x^2 + x - 10$
   (1) It has ∪ shape since it has positive x^2 term
   (2) Intercepts: <u>x-intercepts</u> : y = 0 , $2x^2 + x - 10 = 0$, $x = \frac{-1 \pm \sqrt{81}}{4}$
   (2,0) ; (-5/2,0)
   <u>y-intercept:</u> x = 0 ⇒ y = -10 : (0 , -10)
   (3) The minimum : y' = 4x + 1 = 0 ⇒ x = - 1/4 ⇒ y = -81/8 ⇒ V(-1/4 , -81/8)
   OR, $x = \frac{-b}{2a} = \frac{-1}{4} \Rightarrow$ y = -81/8 ⇒ V(-1/4 , -81/8)

   $g(x) = -3x^2 - 4x + 7$
   (1) It has ∩ shape since it has negative x^2 term
   (2) Intercepts: <u>x-intercepts</u> : y = 0 , $-3x^2 - 4x + 7 = 0$, x = 1 , x = - 7/3
   (1,0) ; (-7/3,0)
   <u>y-intercept:</u> x = 0 ⇒ y = 7 : (0 , 7)
   (3) The maximum : y' = -6x - 4 = 0 ⇒ x = - 2/3 ⇒ y = 25/3 ⇒ V(-2/3 , 25/3)
   OR, $x = \frac{-b}{2a} = \frac{-(-4)}{-6} = \frac{-2}{3} \Rightarrow$ y = 25/3 ⇒ V(-2/3 , 25/3)

Points of intersection: f(x) = g(x)

$$2x^2 + x - 10 = -3x^2 - 4x + 7 \Rightarrow 5x^2 + 5x - 17 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{365}}{10}$$

Graphs:



3. Use a matrix method to find the numbers x, y, z that satisfy
$x - y + z = 4$, $2x + y - z = 2$ and $-x + 4y + 3z = -3$.

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & -1 & 2 \\ -1 & 4 & 3 & -3 \end{bmatrix} \xrightarrow{2I_1 - I_2} \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & -3 & 3 & 6 \\ -1 & 4 & 3 & -3 \end{bmatrix} \xrightarrow{I_1 + I_3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & -3 & 3 & 6 \\ 0 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{I_2 + I_3} \begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 7 & 7 \end{bmatrix}$$

$\therefore\ 7z = 7 \Rightarrow z = 1$

$-3y + 3z = 6 \Rightarrow -3y + 3(1) = 6 \Rightarrow -3y = 6\text{-}3 = 3 \Rightarrow y = -1$

$x - y + z = 4 \Rightarrow x + 1 + 1 = 4 \Rightarrow x = 2$

$(x, y, z) = (2, -1, 1)$

4. Determine the integral : $\int \frac{\sqrt{x}}{\sqrt{\sqrt{x}+1}}dx$

$$t^2 = \sqrt{x}+1 \Rightarrow 2tdt = \frac{1}{2\sqrt{x}}dx \Rightarrow dx = 4t\sqrt{x}dt$$

Substituting : $\int \frac{\sqrt{x}}{\sqrt{\sqrt{x}+1}}dx = \int \frac{\sqrt{x}}{\sqrt{t^2}} \times 4t\sqrt{x}dt = \int \frac{4x}{t} \times tdt = 4\int xdt$

Now $t^2 = \sqrt{x}+1 \Rightarrow \sqrt{x} = t^2 - 1 \Rightarrow x = (t^2-1)^2 = t^4 - 2t^2 + 1$

$$4\int xdt = 4\int (t^4 - 2t^2 + 1)dt = \frac{4t^5}{5} - \frac{2t^3}{3} + t + C$$

$$= \frac{4(\sqrt{x}+1)^5}{5} - \frac{2(\sqrt{x}+1)^3}{3} + \sqrt{x+1} + C$$

5. Use the Lagrange multiplier method to find the maximum value, V , of the function $f(x, y) = xy^{3/2}$ among all combinations of x, y satisfying $x+2y = M$, where M is a positive number. Determine the derivative of V with respect to M and show that this is equal to the value of the Lagrange multiplier.

Lagrange : $L = f - \lambda g = xy^{3/2} - \lambda(x+2y-M)$

$L_x = y^{3/2} - \lambda = 0 \Rightarrow \lambda = y^{3/2} = \sqrt{y^3}$

$L_y = \frac{3}{2}xy^{1/2} - 2\lambda = 0 \Rightarrow \lambda = \frac{3}{4}xy^{1/2} = \frac{3}{4}x\sqrt{y}$

$$\lambda = \lambda \Rightarrow \sqrt{y^3} = \frac{3}{4}x\sqrt{y} \Rightarrow y^3 = \frac{9}{16}x^2y \Rightarrow \frac{y^3}{x^2y} = \frac{9}{16}$$

$\frac{y^2}{x^2} = \frac{9}{16} \Rightarrow 16y^2 = 9x^2$ or $4y = 3x \Rightarrow y = (3/4)x$

substitute for y in g : $x+2y = M$

$y = (3/4)x \Rightarrow x + 2(3/4)x = M \Rightarrow x = 2M/5 \Rightarrow y = (3/4)(2M/5) = 3M/10$

$(x,y) = (2M/5, 3M/10)$

The maximum value : $V = f(2M/5, 3M/10) = (2M/5)(3M/10)^{3/2}$

$\Rightarrow V = \frac{2M}{5} \times \frac{3^{3/2}M^{3/2}}{10^{3/2}} = \frac{2 \times 3^{3/2}}{5 \times 10^{3/2}}M^{1+3/2} = \frac{2 \times 3^{3/2}}{5 \times 10^{3/2}}M^{5/2}$

$$\frac{dV}{dM} = \frac{5}{2} \times \frac{2 \times 3^{3/2}}{5 \times 10^{3/2}}M^{3/2} = \frac{3^{3/2}}{10^{3/2}}M^{3/2} = \left(\frac{3M}{10}\right)^{3/2}$$

Since $\lambda = y^{3/2} = \left(\frac{3M}{10}\right)^{3/2}$, Hence $\lambda = \frac{dV}{dM}$

6. The function f is given by $f(x, y) = x^3 - y^3 - 2xy + 1$. Determine the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Show that the points (0, 0) and (−2/3, 2/3) are critical (or stationary) points of f. Determine whether each of these points is a local maximum, local minimum, or saddle point.

$f_x = 3x^2 - 2y = 0$ and $f_y = -3y^2 - 2x = 0 \Rightarrow x = -3y^2/2$ substitute this in f_x

$\Rightarrow 3(-3y^2/2)^2 - 2y = 0 \Rightarrow 27y^4/4 - 2y = 0 \Rightarrow 27y^4 - 8y = 0$

$\Rightarrow y(27y^3 - 8) = 0 \Rightarrow y = 0$ or $27y^3 - 8 = 0 \Rightarrow y^3 = 8/27 \Rightarrow y = 2/3$

For $y = 0 \Rightarrow x = -3y^2/2 = 0$, the first point is (0,0)

For $y = 2/3 \Rightarrow x = -3y^2/2 = -3(2/3)^2/2 = -12/18 = -2/3$, the second point is (-2/3,2/3)

Nature: $f_{xx} = 6x$, $f_{yy} = -6y$, $f_{xy} = -2$

At (0,0) : $(f_{xx})(f_{yy}) - (f_{xy})^2 = 0 - 4 < 0 \Rightarrow$ (0,0) is a saddle point.

At (-2/3,2/3) : $(f_{xx})(f_{yy}) - (f_{xy})^2 = (-4)(-4) - 4 = 12 > 0$, since $f_{xx} = 6x = 6(-2/3) = -4 < 0$

(-2/3,2/3) maximizes f.

7. A sequence of numbers x_t is constructed as follows: $x_0 = 1$ and every other number in the sequence is obtained by multiplying the previous number by 3 and then subtracting 3. Find a formula, in terms of t and in as simple a form as possible, for x_t.

$x_t = 3x_{t-1} - 3$
$x_1 = 3x_0 - 3$
$x_2 = 3x_1 - 3 = 3(3x_0 - 3) - 3 = 3^2x_0 - 3^2 - 3^1 = 3^2x_0 - (3^1 + 3^2)$
$x_3 = 3x_2 - 3 = 3(3^2x_0 - 3^2 - 3) - 3 = 3^3x_0 - 3^3 - 3^2 - 3^1 = 3^3x_0 - (3^1 + 3^2 + 3^3)$

$x_t = 3^tx_0 - (3^1 + 3^2 + 3^3 + \dots\dots + 3^t)$

$3^1 + 3^2 + 3^3 + \dots\dots + 3^t$ is a G.P. of first term a = 3 and of common ratio r = 3

$$S_t = a \times \frac{r^t - 1}{r - 1} = 3 \times \frac{3^t - 1}{2}$$

$$x_t = 3^tx_0 - 3 \times \frac{3^t - 1}{2} = 3^t - \frac{3}{2}(3^t - 1) \text{ since } x_0 = 1$$

SECTION B
Answer two questions from this section (20 marks each).

8. (a) Find the critical points of the function f(x) = $(1-x)e^{-2x^2}$. For each, determine whether it is a local maximum, a local minimum, or a point of inflexion.

f(x) is of the form : u × v :
u = 1- x ⇒ u' = - 1
v = e^{-2x^2} ⇒ v' = -4x e^{-2x^2}
f'(x) = u'v + v'u = -e^{-2x^2} + (-4x e^{-2x^2})(1-x) = 0
⇒ [- 1 -4x(1-x)] e^{-2x^2} = 0 ⇒ - 1 -4x(1-x) = 0 ⇒ $4x^2$ -4x – 1 = 0
$x = \frac{4 \pm \sqrt{32}}{8} = \frac{4 \pm 4\sqrt{2}}{8} = \frac{1 \pm \sqrt{2}}{2}$ substitute these in f to get the values of y.

f''(x) = (8x-4) e^{-2x^2} + (-4x e^{-2x^2})($4x^2$ – 4x – 1) substitute $\frac{1 \pm \sqrt{2}}{2}$ in f''(x) , if the result is negative you say maximum , if positive you say minimum.

(b) Determine the following integral: $\int e^{\sin x} \sin 2x dx = \int e^{\sin x}(2\sin x \cos x) dx$

Let t = sinx ⇒ dt = cosx dx , substituting :
$\int e^{\sin x}(2\sin x \cos x) dx = 2\int te^t dt$ = 2($te^t - e^t$) + C = 2(sinx $e^{\sin x}$ – $e^{\sin x}$) + c
(Remember $\int xe^x dx = xe^x - e^x + c$ using integration by parts).

(c) A firm is the only producer of two goods, X and Y . The demand equations for X and Y are given by x = 25 − ½ p_X , y = 30 − p_Y , where x and y are the quantities of X and Y demanded (respectively) and p_X, p_Y are (respectively) the prices of X and Y . The firm's joint total cost function (that is, the cost of producing x of X and y of Y) is $x^2 + 2xy + y^2 + 20$. Find an expression in terms of x and y for the profit function. Determine the quantities x and y that maximize the profit.

x = 25 − ½ p_X ⇒ p_X = 50 – 2x
y = 30 − p_Y ⇒ p_Y = 30 – y
Total Revenue = xPx + yPy = x(50-2x) + y(30-y) = $-2x^2 + 50x - y^2 + 30y$
π = TR – TC = $-2x^2 + 50x - y^2 + 30y - x^2 - 2xy - y^2 - 20 = -3x^2 + 50x - 2xy - 2y^2 + 30y - 20$
π_x = -6x + 50 -2y = 0 , π_y = -2x -4y +30 = 0, dividing both sides by -2
⇒ x + 2y - 15 = 0 , Solving simultaneously for x and y : x = 7 , y = 4
$\pi_{xx} = -6 , \pi_{yy} = -4 , \pi_{xy} = -2 \Rightarrow (\pi_{xx})(\pi_{yy}) - (\pi_{xy})^2 = (-6)(-4) - (-2)^2 = 20 > 0$
and since $\pi_{xx} = -6 < 0$ ⇒ (7 , 4) maximizes the profit.

9. (a) A country is in recession and people are becoming newly unemployed at a current rate of 1000 each week. The Government wants to predict how unemployment will grow and there are two different models. In the first model, the rate at which people are becoming newly unemployed increases, with the number becoming unemployed in any particular week being 100 more than the number of those who became unemployed in the previous week. (So, in week 1, the starting week, the number becoming unemployed is 1000, and in week 2 it is 1100, and in week 3 it is 1200, and so on.)
In the second model, the rate at which people are becoming newly unemployed also increases, with the number becoming unemployed in any particular week being 1% more than the number of those who become unemployed in the previous week. (So, in week 1, the starting week, the number becoming unemployed is 1000, and in week 2, it is 1010, and in week 3 it is 1020.1, and so on.)
Find a formula, for each of the two models, for the total number of individuals who become unemployed at some point in the first N weeks. (Your answer will depend on N.)

Model 1 : 1000 , 1100 , 1200 , …….. is an A.P. of first term a = 1000 , and common difference d = 100

$$S_N = \frac{N}{2}\left[2a+(N-1)d\right] = \frac{N}{2}\left[2000+(N-1)(100)\right] = \frac{N}{2}\left[100N+1900\right]$$

Model 2 : Week1 : 1000 , Week2 : 1000 + 1000(0.01) = 1000(1.01) = 1010

Week3 : $1000(1.01)^2 = 1020.1$

1000 , 1000(1.01) , 1000(1.01)](1.01) , …… is a GP of first term a = 1000 and common ratio r = 1.01

$$S_N = a\times\frac{r^N-1}{r-1} = 1000\times\frac{(1.01)^N-1}{1.01-1} = 100\,000\,[(1.01)^N-1]$$

(b) A function f(x) takes the form

$$f(x) = \frac{a}{x^2}+\frac{b}{x}+cx$$

for some constants a, b, c. Given that f(1) = 55, f(2) = 20 and f(3) = 13, find a system of linear equations for a, b, c. By using a matrix method to solve this system, find a, b and c.

f(1) = 55 $\Rightarrow$ a + b + c = 55

f(2) = 20 $\Rightarrow$ a/4 + b/2 + 2c = 20 $\Rightarrow$ a + 2b + 8c = 80

f(3) = 13 $\Rightarrow$ a/9 + b/3 + 3c = 13 $\Rightarrow$ a + 3b + 27c = 117

$$\begin{bmatrix}1&1&1&55\\1&2&8&80\\1&3&27&117\end{bmatrix}\xrightarrow{l_2-l_1}\begin{bmatrix}1&1&1&55\\0&1&7&25\\1&3&27&117\end{bmatrix}\xrightarrow{l_3-l_1}$$

$$\begin{bmatrix}1&1&1&55\\0&1&7&25\\0&2&26&62\end{bmatrix}\xrightarrow{l_3-2l_2}\begin{bmatrix}1&1&1&55\\0&1&7&25\\0&0&12&12\end{bmatrix}$$

$\therefore$ 12c = 12 $\Rightarrow$ c = 1

b + 7c = 25 $\Rightarrow$ b = 25 - 7= 18

a + b + c = 55 $\Rightarrow$ a + 18 + 1 = 55 $\Rightarrow$ a = 36

(a , b , c) = (36 , 18 , 1)

10. (a) The function f(x, y) is given by f(x, y) = $2^{(\ln x)^2 y}$ Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

$f = a^u \Rightarrow f' = u'(\ln a)a^u$, $u = (\ln x)^2 y \Rightarrow u' = 2(\ln x)(1/x)y$

$$\frac{\partial f}{\partial x} = 2(\frac{\ln x}{x})y\ (\mathrm{Ln}2)\ 2^{(\ln x)^2 y}$$

$$\frac{\partial f}{\partial y} = (\ln x)^2(\mathrm{Ln}2)\,2^{x^2 y}$$

(b) A firm has (weekly) production function given by
$q(k, l) = k^{1/4}l^{1/4}$, where k and l denote, respectively, the capital and labour employed.
Each unit of capital costs \$1 a week and each unit of labour costs \$16 a week. Suppose that, when producing any given amount, the firm minimises its total expenditure on capital and labour. Show that when the weekly production level is q, this minimum total expenditure on capital and labour is $8q^2$ per week.
Suppose that the firm pays 1 dollar in all other variable costs (raw materials, and so on) for each unit produced, and that the selling price of the good is fixed at 33 dollars per unit. Find the weekly level of production q that maximises the firm's profit.

Minimise $k + 16\,l$ subject to $k^{1/4}l^{1/4} = q$

$$L = f - \lambda g = k + 16l - \lambda(k^{1/4}l^{1/4} - q)$$

$$\frac{\partial L}{\partial k} = 1 - (\lambda/4)k^{-3/4}l^{1/4} = 0 \Rightarrow \lambda = 4k^{3/4}l^{-1/4}$$

$$\frac{\partial L}{\partial l} = 16 - (\lambda/4)k^{1/4}l^{-3/4} = 0 \Rightarrow \lambda = 64k^{-1/4}l^{3/4}$$

$$\lambda = \lambda \Rightarrow 64k^{-1/4}l^{3/4} = 4k^{3/4}l^{-1/4} \Rightarrow \frac{16k^{-1/4}l^{3/4}}{k^{3/4}l^{-1/4}} = 1$$

$$\Rightarrow 16k^{-1/4}l^{3/4}k^{-3/4}l^{1/4} = 1 \Rightarrow 16k^{-1}l^{1} = 1 \Rightarrow \frac{16\,l}{k} = 1 \Rightarrow k = 16l$$

Substitute this in $k^{1/4}l^{1/4} = q \Rightarrow (16l)^{1/4}l^{1/4} = q \Rightarrow 16^{1/4}l^{1/2} = q \Rightarrow l^{1/2} = q/16^{1/4}$

Squaring : $l = \frac{q^2}{16^{1/2}} = \frac{q^2}{4}$, with $k = 16l = 16(q^2/4) = 4q^2$

minimum expenditure : $k + 16l = 4q^2 + 16(q^2/4) = 8q^2$

TR = pq = 33q , TC = $8q^2 + (1)q = 8q^2 + q$

Profit = π = TR – TC = $33q - 8q^2 - q = -8q^2 + 32q$, $\frac{d\pi}{dq} = -16q + 32 = 0 \Rightarrow q = 2$

11.(a) Find and classify the critical points of the function $f(x, y) = x + 2x^2 + 2xy^2 + y^4$.

$f_x = 1+4x+2y^2 = 0 \Rightarrow x = \frac{-2y^2 - 1}{4}$; $f_y = 4xy + 4y^3 = 0$

$\Rightarrow 4(\frac{-2y^2-1}{4})y + 4y^3 = 0 \Rightarrow (-2y^2-1)y + 4y^3 = 0$

$\Rightarrow y(-2y^2-1+4y^2) = 0 \Rightarrow y(2y^2-1) = 0 \Rightarrow y = 0$ or $y = \pm 1/\sqrt{2}$

For $y = 0 \Rightarrow x = -1/4$, for $y = \pm 1/\sqrt{2} \Rightarrow x = -1/2$

The critical points are : (-1/4 , 0) , $(-1/2, -1/\sqrt{2})$, $(-1/2, 1/\sqrt{2})$

To check whether it maximises or minimises f or a saddle point: $(f_{xx})(f_{yy}) - f_{xy}^2$

$f_{xx} = 4$, $f_{yy} = 4x + 12y^2$; $f_{xy} = 4y$

at (-1/4 , 0): $f_{xx} = \mathbf{4}$, $f_{yy} = 4x + 12y^2 = 4(-1/4) + 12(0) = \mathbf{-1}$, $f_{12} = 4y = 4(0) = \mathbf{0}$

$(f_{xx})(f_{yy}) - f_{12}^2 = (4)(-1) - 0 = -4 < 0 \Rightarrow$ (-1/4 , 0) is a Saddle point.

at $(-1/2, -1/\sqrt{2})$: $f_{xx} = 4$, $f_{yy} = 4$; $f_{xy} = -4/\sqrt{2}$

$(f_{xx})(f_{yy}) - f_{xy}^2 = 8 > 0$ Since $f_{xx} = \mathbf{4 > 0}$ **;** $(-1/2, -1/\sqrt{2})$ minimises f .

at $(-1/2, 1/\sqrt{2})$: $f_{xx} = 4$, $f_{yy} = 4$; $f_{xy} = 4/\sqrt{2} \Rightarrow (f_{xx})(f_{yy}) - f_{xy}^2 = 8 > 0$

Since $f_{xx} = \mathbf{4 > 0}$ **;** $(-1/2, 1/\sqrt{2})$ minimises f

(b) An investor saves money in a bank account paying interest at a fixed rate of 4%, where the interest is paid once per year, at the end of the year. He deposits an amount L at the beginning of each year for n years. Show that he will then have saved an amount equal to $25L((1.04)^n - 1)$ immediately after his last deposit.

At the beginning of year 1 : L

At the beginning of year 2 :

$P_2 = (L)(1+r) + L$

At the beginning of year 3 :

$P_3 = P_2(1+r) + L$

$= [(L)(1+r) + L](1+r) + L$

$P_3 = L(1+r)^2 + L(1+r) + L$

At the beginning of year 4:

$P_4 = P_3(1+r) + L$

$= [L(1+r)^2 + L(1+r) + L](1+r) + L$

$P_4 = L(1+r)^3 + L(1+r)^2 + L(1+r) + L$

At the beginning of the Nth year :

$P_N = L(1+r)^{N-1} + L(1+r)^{N-2} + L(1+r)^{N-3} + \ldots + L(1+r)^2 + L(1+r) + L$

$P_N = L[(1+r)^{N-1} + (1+r)^{N-2} + \ldots\ldots + (1+r)^2 + (1+r) + (1+r)^0]$

Now : $(1+r)^0 + (1+r) + (1+r)^2 + \ldots\ldots\ldots + (1+r)^{N-1}$ is

a Geometric Progression of first term 1 , common ratio (1+r)

and number of terms = N

Its sum is : $1 \times \dfrac{(1+r)^N - 1}{1 + r - 1} = \dfrac{(1+r)^N - 1}{r}$

$\Rightarrow P_N = L\left(\dfrac{(1+r)^N - 1}{r}\right)$ with $r = 0.04$, $P_N = L\left(\dfrac{(1.04)^N - 1}{0.04}\right) = 25L((1.04)^n - 1)$

(c) Show that the sum of the first n positive even numbers is $n^2 + n$.

The Arithmetic progression : 2 , 4 , 6 , 8 , ……….

a = 2 , d = 2

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[4 + (n-1)(2)] = \frac{n}{2}[2n + 2] = n^2 + n$$

END OF PAPER