International Institute for
Technology and Management

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Unit 05a: Mathematics 1 – (MathB) Assignment – 2

1. Determine the derivatives of the following functions : **(18 Marks)**

a. $y = (3e^{2x} - \cos x)(\sin x - 3)^2$ of the form uxv

$u = 3e^{2x} - \cos x \Rightarrow u' = 6e^{2x} - (-\sin x) = 6e^{2x} + \cos x$

$v = (\sin x - 3)^2 \Rightarrow v' = 2(\sin x - 3)(\cos x)$

$y' = u'v + v'u = (6e^{2x} + \cos x)(\sin x - 3)^2 + 2(\sin x - 3)(\cos x)(3e^{2x} - \cos x)$

b. $y = \dfrac{\sqrt{1+x^2}}{1+e^x}$ of the form u/v

$u = \sqrt{1+x^2} \Rightarrow u' = \dfrac{2x}{2\sqrt{1+x^2}} = \dfrac{x}{\sqrt{1+x^2}}$

$v = 1+e^x \Rightarrow v' = e^x$

$$y' = \frac{u'v - v'u}{v^2} = \frac{\left(\frac{x}{\sqrt{1+x^2}}\right)(1+e^x) - e^x\left(\sqrt{1+x^2}\right)}{(1+e^x)^2}$$

c. $y = \ln\left(\dfrac{e^x - 1}{e^x + 1}\right) = \ln(e^x - 1) - \ln(e^x + 1)$

$$y' = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{e^x(e^x+1) - e^x(e^x - 1)}{e^{2x} - 1} = \frac{2e^x}{e^{2x} - 1}$$

d. $y = \left(x^2 + \dfrac{1}{x}\right)[5 + \ln(x + e^x)]$ of the form uxv

$u = x^2 + \dfrac{1}{x} \Rightarrow u' = 2x - \dfrac{1}{x^2}$

$v = 5 + \ln(x + e^x) \Rightarrow v' = \dfrac{1+e^x}{x+e^x}$

$$y' = u'v + v'u = \left(2x - \frac{1}{x^2}\right)[5 + \ln(x+e^x)] + \left(\frac{1+e^x}{x+e^x}\right)\left(x^2 + \frac{1}{x}\right)$$

e. $y = \dfrac{\sin x}{1 - \cos x}$ of the form u/v u = sinx $\Rightarrow$ u' = cosx , v= 1- cosx $\Rightarrow$ v' = -(-sinx) = sinx

$$y' = \frac{u'v - v'u}{v^2} = \frac{(\cos x)(1-\cos x) - \sin x(\sin x)}{(1-\cos x)^2}$$

$$y' = \frac{\cos x - \cos^2 x - \sin^2 x}{(1-\cos x)^2} = \frac{\cos x - (\cos^2 x + \sin^2 x)}{(1-\cos x)^2}$$

$$= \frac{\cos x - 1}{(1-\cos x)^2} = \frac{-(1-\cos x)}{(1-\cos x)^2} = \frac{-1}{1-\cos x}$$

f. **y =** $(x^2 - x + 3)e^{\frac{-x^2}{2}}$ of the form uxv

u = x^2 – x + 3 $\Rightarrow$ u' = 2x – 3

v= $e^{\frac{-x^2}{2}}$ $\Rightarrow$ v' = (-2x/2) $e^{\frac{-x^2}{2}}$ = -x$e^{\frac{-x^2}{2}}$

y' = u'v+v'u = (2x-3) $e^{\frac{-x^2}{2}}$ + (-x$e^{\frac{-x^2}{2}}$)(x^2 – x + 3)

y' = $e^{\frac{-x^2}{2}}$ (2x -3 – x^3 +x^2 -3x) = (– x^3 +x^2 –x -3) $e^{\frac{-x^2}{2}}$

2. Find all stationary points of the following functions and determine whether they are maxima,minima or inflection points: **(12 Marks)**

a. **y = 3x^5 – 25x^3 + 60x** $\Rightarrow$ y' = 15x^4 -75x^2 + 60 = 0
let t = $x^2 \geq 0$ $\Rightarrow$ 15t^2 – 75t + 60 = 0
a + b + c = 0 $\Rightarrow$ t = 1 or t = c/a = 60/15 = 4
but t = x^2 $\Rightarrow$ x^2 = 1 $\Rightarrow$ x = ± 1 or x^2 = 4 $\Rightarrow$ x = ± 2
We have four critical points:
x = -1 $\Rightarrow$ y= - 38 , 1st point : (-1,-38)
x = 1 $\Rightarrow$ y= 38 , 2nd point : (1,38)
x = -2 $\Rightarrow$ y= - 16 , 3rd point : (-2,-16)
x=2 $\Rightarrow$ y = 16 , 4th point : (2 , 16)
Second derivative test: **y" = 60x^3 – 150x**
<u>at (-1 , -38) :</u> y"(-1) = 90 >0 $\Rightarrow$ x = -1 minimizes the function and the value of the minimum is - 38.
<u>at (1 , 38) :</u> y"(1) = - 90 < 0 $\Rightarrow$ x = 1 maximizes the function and the value of the maximum is 38.
<u>at (-2 , -16) :</u> y"(-2) = -180 < 0 $\Rightarrow$ x = -1 maximizes the function and the value of the maximum is -16.
<u>at (2 , 16) :</u> y"(2) = 180 >0 $\Rightarrow$ x = 2 minimizes the function and the value of the minimum is 16.

b. **$y = (x^2 - x - 1)e^{-x}$** $\Rightarrow$ $y' = (2x-1)e^{-x} - (x^2 - x - 1)e^{-x}$

$y' = e^{-x}(2x - 1 - x^2 + x + 1) = e^{-x}(-x^2 + 3x)$

but $e^{-x} \neq 0 \Rightarrow -x^2 + 3x = 0 \Rightarrow -x(x-3) = 0$

$\Rightarrow x = 0$ or $x = 3$

$x = 0 \Rightarrow y = -1$, 1st point : $(0,-1)$

$x = 3 \Rightarrow y = 5e^{-3}$, 2nd point : $(3, 5e^{-3})$

Second derivative test; $y' = (-2x+3)e^{-x} - (-x^2+3x)e^{-x}$

$y' = (x^2 - 5x + 3)e^{-x}$

<u>at $(0,-1)$</u> : $y''(0) = 3 > 0 \Rightarrow x = 0$ minimizes the function and the value of the minimum is -1.

<u>at $(3,5e^{-3})$</u> : $y''(3) = -3e^{-3} < 0 \Rightarrow x = 3$ maximizes the function and the value of the maximum is $5e^{-3}$.

c. **$y = \dfrac{\ln x}{x^2}$, defined for $x > 0$**

y' = $\dfrac{(1/x)(x^2) - 2x\ln x}{x^4} = \dfrac{x - 2x\ln x}{x^4} = 0$

$x - 2x\ln x = 0$ $\Rightarrow x(1 - 2\ln x) = 0$ but $x > 0$

$\Rightarrow 1 - 2\ln x = 0 \Rightarrow$ **$\ln x = ½ \Rightarrow x = e^{1/2}$** $= \sqrt{e}$

(Remember : $\ln x = a \Rightarrow x = e^a$)

$x = e^{1/2} \Rightarrow$ **y =** $\dfrac{\ln e^{1/2}}{(e^{1/2})^2} = \dfrac{1/2}{e} = \dfrac{1}{2e}$

The critical point is $(e^{1/2}, 1/2e)$

Second derivative test : $y' = \dfrac{x(1 - 2\ln x)}{x^4} = \dfrac{1 - 2\ln x}{x^3}$

y'' = $\dfrac{(-2/x)(x^3) - 3x^2(1 - 2\ln x)}{x^6} = \dfrac{-2x^2 - 3x^2(1 - 2\ln x)}{x^6}$

<u>at $(e^{1/2}, 1/2e)$</u> : $y''(e^{1/2}) = (-2e - 0)/e^3 = -2/e^2 < 0$

since $1 - 2\ln x = 1 - 2\ln e^{1/2} = 1 - 2(1/2)\ln e = 1 - 1 = 0$, $(\ln e = 1)$

$\Rightarrow x = e^{1/2}$ maximizes the function and the value of the maximum is $1/2e$.

3. Evaluate the following integrals: **(12 Marks)**

a. $\displaystyle\int \frac{\sqrt{1 - \ln x}}{x}\,dx$ **$u = 1 - \ln x$** $\Rightarrow du = -dx/x$

$= \int -\sqrt{u}\,du = -\int u^{1/2}du$ **= -** $\dfrac{u^{3/2}}{3/2}$ **+ C = -** $\dfrac{2(1 - \ln x)^{3/2}}{3}$ **+C**

b. $\int \frac{3x-3}{x^2-2x+3}dx$ **u = x² -2x + 3** $\Rightarrow$ du = (2x -2) dx

dx = du/2x-2

$= \int \frac{3x-3}{u}\frac{du}{2x-2}$ **=** $\int \frac{3(x-1)}{u}\frac{du}{2(x-1)}$ **=** $\frac{3}{2}\int \frac{du}{u}$

= $\frac{3}{2}$ ln | u | + C = $\frac{3}{2}$ ln | **x^2 -2x + 3** | + C

c. $\int \sin x \cos^3 x dx$ **u = cosx** $\Rightarrow$ du = - sinx dx

= $\int -u^3 du$ **= $-u^4/4$ + C = $-\cos^4 x/4$ + C**

d. $\int \frac{dx}{e^x(e^{-x}+1)^2}$ **u = e^{-x} + 1** $\Rightarrow$ du = - e^{-x} dx

$= \int \frac{e^{-x}dx}{(e^{-x}+1)^2}$ **=** $\int \frac{-du}{u^2}$ **= -** $\int u^{-2}$du **= 1/u + C =** $\frac{1}{e^{-x}+1}+C$

4. A firm's marginal cost function is: **(8 Marks)**

$$\frac{20}{\sqrt{q}}e^{\sqrt{q}}+3q^2+\frac{q}{(q^2+1)^2}$$ and the cost of producing 4 units is $10e^2$.

Determine the total cost function.

TC = $\int MCdq$ = $\int \frac{20}{\sqrt{q}}e^{\sqrt{q}}+3q^2+\frac{q}{(q^2+1)^2}$ for the first one use u= $\sqrt{q}$

For the last one , use u = q^2 + 1

TC = $10e^{\sqrt{q}}$ + q^3 + ½ ($1/q^2$+1) + C

Now TC (4) = $10e^2$ $\Rightarrow$ $10e^{\sqrt{4}}$ + 4^3 + ½ ($1/4^2$+1) + C = $10e^2$

$\Rightarrow$ 64 + ½ (1/16+1) + C = 0 $\Rightarrow$ C = -64 - 17/32 = - 2065/32

TC = $10e^{\sqrt{q}}$ + q^3 + ½ ($1/q^2$+1) - 2065/32

<u>END of ANSWERS</u>