05a Mathematics 1
Examiners' Report 2006
Zone A

General remarks

Showing your working

We start by emphasising that candidates should *always* include their working. This means two things. First, they should not simply write down the answer in the examination script, but explain the method by which it is obtained. Secondly, they should include rough working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined.

We also stress that if a student has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Covering the syllabus

Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, *any* topic could potentially appear in Section A.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the student does indeed know the methods, understands them, and is able to use them, and not merely whether they vaguely remember them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different from previous years' questions. You should *expect* to be surprised by some of the questions. Of course, candidates will only be examined on material in the syllabus, so all questions can be answered using the material of the subject. There will be enough, routine, familiar content in the examination so that a student who has achieved competence in the subject will pass, but, of course, for a high mark, more is expected: students will have to demonstrate an ability to solve new and unfamiliar problems.

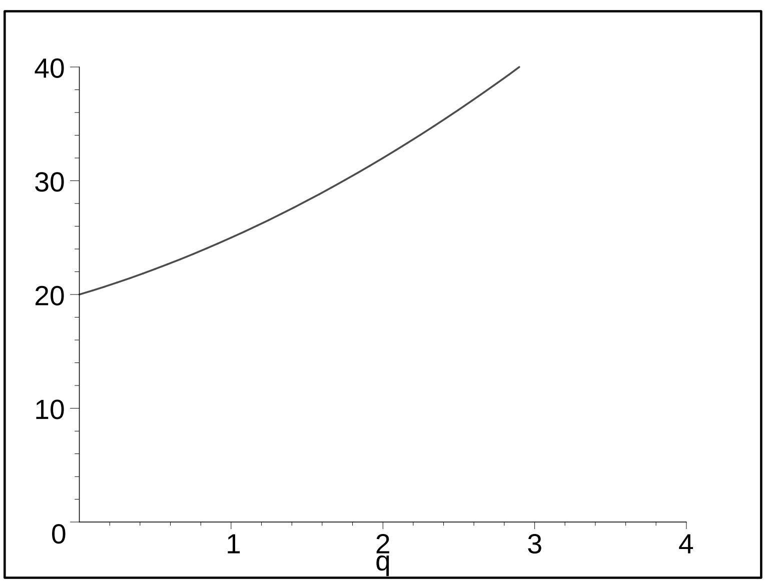
Students are reminded that calculators are *not* permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.

Specific comments on questions

1. This was a straightforward and standard question, very well done in general. In sketching the demand curve, $p = q^2 + 4q + 20$, it is useful to obtain some additional information. We know that it is a parabola, because the equation is quadratic. We can see that $q^2 + 4q + 20 = (q + 2)^2 + 16$, which shows that it always lies above the q-axis and that its minimum value is at $q = -2$. (The same conclusions can be drawn by noting that the equation has no roots and by finding that the derivative is 0 at -2.) The graph should also indicate the value if p at which the graph crosses the p-axis (which is $p = 20$). The graph is strange (but not impossible) for a demand curve, in that it slopes upwards. The graph is shown below.

For the equilibrium, we need to solve $q^2 + 4q + 20 = -q^2 - 10q + 176$. This can be written as a quadratic equation in standard form as $2q^2 + 14q - 156 = 0$. This can be solved by factorising or using the formula for the solutions of a quadratic. The solutions are $q = -13$ and $q = 6$, so the solution we seek (the economically meaningful one) is $q = 6$. The corresponding value of p is $p = 80$.

The curve should *not* be 'sketched' by plotting points: that is 'plotting' not 'sketching'.



2. This question was quite straightforward and well done on the whole, though a few candidates misunderstood the information given and, as a result, produced the wrong profit function. From the information given, we can find that $p = 4/\sqrt{q}$, so the total revenue is $TR = (4/\sqrt{q})q = 4\sqrt{q}$ and the profit is $\Pi = TR - TC = 4\sqrt{q} - q^2$, noting that the total cost function is q^2. We then solve $\Pi'(q) = 0$, which is $\frac{4}{2\sqrt{q}} - 2q = 0$, leading to $q\sqrt{q} = 1$, and hence $q = 1$. Corresponding to this, we have $p = 4$. Furthermore, we should check that this does indeed maximise the profit. We have $\Pi''(q) = -1/q^{3/2} - 2$ so $\Pi''(1) < 0$ and hence it's a maximum.

3. This was a straightforward and standard question. The augmented matrix is

$$\begin{pmatrix} 1 & -2 & 10 & 5 \\ 2 & 1 & -2 & 4 \\ 1 & 3 & 4 & 7 \end{pmatrix}.$$

Using row operations to reduce, we have

$$\begin{pmatrix} 1 & -2 & 10 & 5 \\ 2 & 1 & -2 & 4 \\ 1 & 3 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 10 & 5 \\ 0 & 5 & -22 & -6 \\ 0 & 5 & -6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 10 & 5 \\ 0 & 5 & -22 & -6 \\ 0 & 0 & 16 & 8 \end{pmatrix},$$

so we have $z = 1/2$, $5y = -6 + 11 = 5$, so $y = 1$ and $x = 5 - 5z + 2y = 2$. Some candidates used invalid row operations (eg, multiplying two rows together or subtracting a fixed number from each entry of a row, a surprisingly common error). Be sure that you understand what row operations are allowable.

Some used Cramer's rule. That is perfectly acceptable, but possibly more prone to arithmetical error.

4. This is an easy integral to do once the right substitution is chosen, and that is the key to solving it. If we let $u = 1 + \sqrt{e^{-x}}$, then $du = -\frac{1}{2}e^{-x/2}dx$ and the integral reduces to the very easy $I = -2\int \sqrt{u}\, du$. This gives $= -\frac{4}{3}u^{3/2} + c$, which is $= -\frac{4}{3}(1 + \sqrt{e^{-x}})^{3/2} + c$. (Other substitutions, such as $u = \sqrt{x}$ will also work.)

5. This was badly done. This is an unusual question, but every examination contains unusual questions, and this is a deliberate strategy on the examiners' part to ensure the standard of the examination. (See the discussion in the final section of this report.) However, this is not fundamentally a hard question. All it needs is the chain rule and the product rule.

Let $g = -ax - by$. Then, by the product and chain rules,

$$V_x = U_x e^g + g_x U e^g = U_x e^g - aUe^g.$$

(Here, we use the V_x notation for $\partial V/\partial x$, and so on.) Also,

$$V_y = U_y e^g + g_y U e^g = U_y e^g - bUe^g.$$

Differentiating V_x again with respect to x, and using the product and chain rules, we have

$$V_{xx} = U_{xx}e^g - aU_x e^g - aU_x e^g + a^2Ue^g = U_{xx}e^g - 2aU_x e^g + a^2Ue^g.$$

With $a = 1, b = 4$, we have (with $g = -x - 4y$), $V_x = U_x e^g - Ue^g$ $V_y = U_y e^g - 4Ue^g$, $V_x = U_x e^g - \frac{1}{2}Ue^g$, and $V_x x = U_{xx}e^g - 2U_x e^g + Ue^g$.

We have $V_y = V_{xx} + 2V_x - 3V$, so

$$U_y e^g - 4Ue^g = (U_{xx} - 2U_x + U + 2U_x - 2U - 3U)\, e^g$$

and so, simplifying, $U_y = U_{xx}$, as required.

6. Two-variable optimisation problems are very standard. Finding the critical point in this particular problem required solving some tricky equations, but the usual technique of using the fact that the partial derivatives are 0 to substitute for one variable in terms of the other leads to the answer.

The partial derivatives are

$$f_x = 1 + \frac{2y}{x^2}$$

and

$$f_y = -\frac{4}{y^2} - \frac{2}{x}.$$

We solve $f_x = f_y = 0$. Now, $f_x = 0$ means $y = -(1/2)x^2$. Then, $f_y = 0$ implies $2/(-x^2/2)^2 = -1/x$, so $x^3 = -8$ and $x = -2$. Then, $y = -x^2/2 = -2$.

The second derivatives are

$$f_{xx} = -4y/x^3, \quad f_{yy} = 8/y^3, \quad f_{xy} = 2/x^2.$$

At $(-2, -2)$ we have $f_{xx} = -1, f_{yy} = -1, f_{xy} = 1/2$, so $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, and hence this is a local maximum point.

7. Constrained optimisation problems such as this are standard and routine, and the Lagrange method is the obvious way to tackle them. (Note, though, that the question did not require the use of Lagrange's method, so a method based on substituting for x or y in terms of the other variable using the constraint was acceptable.) This problem is similar to, but substantially easier than, Question 10 (a) of the 2005 ZB and ZA examination papers.

The Lagrangian is $L = (x + 1)^3(y + 1)^2 - \lambda(x + y - 13)$. (Or, if you like, equally acceptably, $L = (x + 1)^3(y + 1)^2 + \lambda(x + y - 13)$. We need to solve

$$L_x = 3(x + 1)^2(y + 1)^2 - \lambda = 0$$

and

$$L_y = 2(x + 1)^3(y + 1) - \lambda = 0$$

simultaneously, alongside the constraint equation $x+y = 13$. These first two equations imply that

$$3(x + 1)^2(y + 1)^2 = 2(x + 1)^3(y + 1),$$

so $3(y+1) = 2(x+1)$ and $y = \frac{2}{3}x - \frac{1}{3}$ (or, alternatively, we could express x in terms of y). The constraint is $x + y = 13$, which means

$$\frac{5}{3}x - \frac{1}{3} = 13$$

and hence $x = 8$ and $y = 5$.

8(a) This is a completely standard type of problem. A very similar problem appeared in the sample examination paper in the old edition of the subject guide (which students taking the paper in 2006 will have used) and in previous examinations (Question 10(a), 2004 ZB and ZA papers). Those who attempted this did well.

The equations are equivalent to

$$\begin{aligned} 2x_1 + 5x_2 + 7x_3 &= 294 \\ 2x_1 + x_2 &= 70 \\ 12x_1 - 2x_2 + 4x_3 &= 44. \end{aligned}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 5 & 7 & 294 \\ 2 & 1 & 0 & 70 \\ 12 & -2 & 4 & 44 \end{pmatrix}$$

Using row operations to reduce, we have

$$\begin{pmatrix} 2 & 5 & 7 & 294 \\ 2 & 1 & 0 & 70 \\ 12 & -2 & 4 & 44 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 70 \\ 2 & 5 & 7 & 294 \\ 6 & -1 & 2 & 22 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 1 & 0 & 70 \\ 0 & 4 & 7 & 224 \\ 0 & -4 & 2 & -188 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 70 \\ 0 & 4 & 7 & 224 \\ 0 & 0 & 9 & 36 \end{pmatrix},$$

so equilibrium prices are $x_3 = 4$, $x_2 = 49$ and $x_1 = 21/2$.

8(b) Some candidates had difficulty with this first integral because they did not know how to work with trigonometrical functions. But such functions are an important part of this subject and are discussed in the subject guide. The easiest way is to make the substitution $u = \tan x$. Then $du = \dfrac{1}{\cos^2 x} dx$. (This can be calculated using the quotient rule, the fact that $\tan x = \sin x / \cos x$ and the fact that $\sin^2 x + \cos^2 x = 1$.)

So the integral is

$$\int_0^1 \frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_0^1 = 2.$$

(Note that we have changed the limits: if you don't do this, you will need to revert back to x once you have integrated: that's perfectly acceptable.) Note that we have used the fact that $\tan \pi/4 = 1$. This is stated explicitly in a table in the subject guide. Students should be comfortable with radians (as explained in the guide) and should know the values of the trigonometrical functions at the key values of $0, \pi/6, \pi/4, \pi/3, \pi/2$: see the table in the subject guide.

For the second integral, the easiest (and most obvious) approach is integration by parts: we have

$$\int (x+2)^2 \ln x \, dx = \frac{1}{3}(x+2)^3 \ln x - \frac{1}{3}\int (x+2)^3 \frac{1}{x}\, dx$$
$$= \frac{1}{3}(x+2)^3 \ln x - \frac{1}{3}\int (x^3 + 6x^2 + 12x + 8)\frac{1}{x}\, dx$$
$$= \frac{1}{3}(x+1)^3 \ln x - \frac{1}{3}\int \left(x^2 + 6x + 12 + \frac{8}{x}\right) dx$$
$$= \frac{1}{3}(x+2)^3 \ln x - \frac{1}{9}x^3 - x^2 - 4x - \frac{8}{3}\ln x + c.$$

9(a) This is a straightforward question. (It is similar to, but easier than, sample question 5 of Chapter 4 in the subject guide: see the solution to that problem and the discussion given there.) All that is required is a recognition that marginal cost is the derivative of total cost, and an ability to use partial fractions. What we need is $TC(2) - TC(1)$, where TC is the total cost function. This is

$$\int_1^2 \frac{q+1}{q^2+6q+8} dq.$$

Now, using partial fractions,

$$\frac{q+1}{q^2+6q+8} = \frac{q+1}{(q+2)(q+4)} = -\frac{1/2}{q+2} + \frac{3/2}{q+4}.$$

So

$$TC(2) - TC(1) = [-(1/2)\ln(q+2) + (3/2)\ln(q+4)]_1^2 = (1/2)\ln(3/4) + (3/2)\ln(6/5).$$

It is incorrect to simply calculate $MC(1)$ and say that the extra cost is equal to this value. Remember that the marginal cost is the derivative of the total cost and it gives an approximation to the increased cost of producing one more unit when many are being produced. But here the increase in production, from 1 to 2, is so large as to make this approximation invalid. We need, to answer the question correctly, to integrate to determine the total cost function.

9(b) We have $P(t) = Ve^{-0.1t} = 10e^{2\sqrt{t}}e^{-0.1t}$, so

$$P'(t) = 10\left(\frac{1}{\sqrt{t}} - 0.1\right)e^{-0.1t}.$$

Therefore, $P'(t) = 0 \iff \sqrt{t} = 10 \iff t = 100$. At $t = 100$, P' changes sign from positive to negative, so it is a maximum. (Alternatively, we could calculate $P''(t)$ and note that $P''(100) < 0$.)

9(c) This problem is hard if you do not follow the very explicit hint given in the question. But if you do what it suggests, it is much easier. You should always follow a hint if one is given. We have

$$f_x = 4x(4xy + 5) + (2x^2 + 2y^2 - 3)4y$$

and

$$f_y = 4y(4xy + 5) + (2x^2 + 2y^2 - 3)4x.$$

Now, $f_x = f_y = 0 \implies f_x + f_y = 0$, so, following the hint, and adding these two equations,

$$f_x + f_y = (4x + 4y)(4xy + 5) + (4x + 4y)(2x^2 + 2y^2 - 6) = 0,$$

so

$$4(x + y)(2x^2 + 4xy + 2y^2 + 2) = 0.$$

This simplifies to

$$4(x + y)(2(x + y)^2 + 2) = 0$$

so we must have $y = -x$. The equation $f_x = 0$ then becomes $4x(-4x^2 + 5) - 4x(4x^2 - 3) = 00$, which is $32x - 32x^3 = 0$, so $x(1 - x^2) = 0$ and $x = 0, 1$ or -1 and $y = 0, -1, 1$, correspondingly. So the critical points are $(0, 0), (1, -1)$ and $(-1, 1)$.

10(a). Problems such as this, although regularly featuring in these examinations and not hard, seem to be problematic for students. This year, there was no question on sequences and series in Section A, but no inference should be drawn from that fact for future years. (See the comments on future examinations.) To answer this question, we have to be clear about what we want to find, and how we will find it. Let y_t be balance in account after t full years. (This is the key thing to monitor.) So $y_0 = L$. Then $y_1 = (1.05)L - D$,

$$y_2 = (1.05)y_1 - D = (1.05)^2L - (1.05)D - D,$$

$$y_3 = (1.05)y_2 - D = (1.05)^3L - (1.05)^2L - (1.05)L - L,$$

and, in general

$$y_t = (1.05)^tL - (1.05)^{t-1}D - (1.05)^{t-2}D - \cdots - D.$$

Now, we must have $y_N \geq 0$ in order to make the required number of withdrawals. So

$$(1.05)^NL - (1.05)^{N-1}D - (1.05)^{N-2}D - \cdots - D \geq 0.$$

This simplifies (noting the geometric progression) to

$$(1.05)^NL - D\frac{1 - (1.05)^N}{1 - 1.05} \geq 0.$$

So

$$L \geq 20D\left(1 - \frac{1}{(1.05)^N}\right).$$

10(b) This is a very standard utility-maximisation problem, with a budget constraint. These occur frequently in these examination papers (e.g., Question 11 of the 2003 ZA paper, Question 5 of the 2003 ZB paper, Question 5 of the 2004 ZA paper, Question 5 of the 2004 ZB paper, Question 10(a) of the 2005 ZA paper, and Question 10(a) of the 2005 ZB paper) though, of course, the precise utility function is different from ones previously used.

The problem, therefore, is the familiar one: maximise $(x^\beta + 3y^\beta)^{1/\beta}$ subject to $x + y = M$. As usual, the Lagrange multiplier approach is best. The Lagrangian is $L = (x^\beta + 3y^\beta)^{1/\beta} - \lambda(x + y - M)$. We solve

$$L_x = x^{\beta-1}(x^\beta + 3y^\beta)^{(1/\beta-1)} - \lambda = 0$$

$$L_y = 3y^{\beta-1}(x^\beta + 3y^\beta)^{(1/\beta-1)} - \lambda = 0$$

together with the constraint $x + y = M$. We solve $L_x = L_y = 0$. From the first two equations, we have $x^{\beta-1} = 3y^{\beta-1}$, from which we have $x = 3^{1/(\beta-1)}y$. The constraint, $x + y = M$, becomes $y(1 + 3^{1/(\beta-1)}) = M$ so $y = M/(1 + 3^{1/(\beta-1)})$ and

$$x = 3^{1/(\beta-1)}y = \frac{3^{1/(\beta-1)}M}{1 + 3^{1/(\beta-1)}}.$$

The maximum u is $(x^\beta + 3y^\beta)^{1/\beta}$ with these values of x and y. This is

$$\frac{M}{1 + 3^{1/(\beta-1)}}(3^{\beta/(\beta-1)} + 3)^{1/\beta}.$$

11(a) This was an exercise in partial differentiation and the manipulation of expressions, a standard type of problem (see Question 9(b), 2003 ZB examination, and Question 9(b) in the mock examination in the old edition of the subject guide). Care is required, so these problems should not be rushed. We have

$$f_x = \frac{1}{2\sqrt{2x + y + \sqrt{4x^2 + y^2}}}\left(2 + \frac{4x}{\sqrt{4x^2 + y^2}}\right),$$

$$f_y = \frac{1}{2\sqrt{2x + y + \sqrt{4x^2 + y^2}}}\left(1 + \frac{y}{\sqrt{4x^2 + y^2}}\right).$$

So

$$\begin{aligned}
xf_x + yf_y &= \frac{1}{2\sqrt{2x + y + \sqrt{4x^2 + y^2}}}\left(2x + \frac{4x^2}{\sqrt{4x^2 + y^2}} + y + \frac{y^2}{\sqrt{4x^2 + y^2}}\right) \\
&= \frac{1}{2\sqrt{2x + y + \sqrt{4x^2 + y^2}}}\left(2x + y + \frac{4x^2 + y^2}{\sqrt{4x^2 + y^2}}\right) \\
&= \frac{1}{2\sqrt{2x + y + \sqrt{4x^2 + y^2}}}\left(2x + y + \sqrt{4x^2 + y^2}\right)
\end{aligned}$$

$$= \frac{1}{2}\sqrt{2x + y + \sqrt{4x^2 + y^2}}$$
$$= \frac{1}{2}f.$$

11(b) Two-commodity maximisation problems are standard problems (2003 ZB, Question 8(b); 2003 ZA, Question 8(b); 2004 ZB, Question 4; 2005 ZB, Question 10(b); 2005 ZA, Question 10(b); and Chapter 5 of the subject guide). What is different here is the presence of the constant c in the expression for the total cost function. But, in all other respects, this is a completely standard question and the standard methods apply.

The profit function is

$$\Pi = xp_X + yp_Y - TC = x(4-2x) + y(2-2y) - (x^2+cxy+y^2) = 4x+2y-3x^2-3y^2-cxy.$$

We solve $\Pi_x = \Pi_y = 0$, which is $4-6x-cy = 0, \;\; 2-6y-cx = 0$. These are equivalent to $36x + 6cy = 24, \;\; c^2x + 6cy = 2c$. Subtracting the second of these equations from the first eliminates y and we find that

$$x = \frac{24-2c}{36-c^2}.$$

(There are other ways of solving the equations.) Because $0 < c < 3$, this value of x is positive. Then,

$$y = \frac{1}{c}(4-6x) = \frac{12-4c}{36-c^2}.$$

Again, note that, because $0 < c < 3$, this value of y is positive. We have

$$\Pi_{xx} = -6, \Pi_{yy} = -6, \Pi_{xy} = -c,$$

so $\Pi_{xx} < 0$ and $\Pi_{xx}\Pi_{yy} - \Pi_{xy}^2 = 36 - c^2 > 0$, since $0 < c < 3$, so it is a maximum.

Examination paper for 2007

There will be no change to the format, style or number of questions in the examination paper for 2007.