05a Mathematics 1
Examiners' Report 2004
Zone A

Exam technique: general remarks

We start by emphasising that candidates should *always* include their working. This means two things. First, they should not simply write down the answer in the exam script, but explain the method by which it is obtained. Secondly, they should include rough working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined.

We also stress that if a candidate has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, *any* topic could potentially appear in Section A.

Students are reminded that calculators are *not* permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.

Specific comments on questions

1. It should be realised that the curves are parabolas, and their intersections with the axes should be indicated. Finding the intersection point involves solving the equation $4x^2 - 8x - 1 = -4x^2 - 2x + 4$ which, rewritten as a quadratic equation in standard form, becomes $8x^2 - 6x - 5 = 0$, and this can be solved in the usual ways. The positive solution is $x = 5/4$.

2. We find $f'(x) = 2e^{-x^2} - 2x(1+2x)e^{-x^2} = 2e^{-x^2}(1-x-2x^2)$. Solving $f'(x) = 0$ means solving $1 - x - 2x^2 = 0$, which has solutions $1/2$ and -1 (as can be seen by factorising or by using the formula for the solutions of a quadratic equation). The nature of each critical point can be checked by examining the behaviour of the derivative around each point or (more difficult in this particular problem) by computing the second derivative at each point. It is found that $x = 1/2$ gives a maximum.

3. This integral can be solved using the partial fractions method. We find that

$$\frac{x}{x^2 - x - 2} = \frac{2/3}{x-2} + \frac{1/3}{x+1}$$

and so the integral is

$$\frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + c.$$

4. We have

$$\frac{\partial f}{\partial x} = 2x - 2y,$$
$$\frac{\partial f}{\partial y} = -2x + 4y - 2.$$

The stationary point of f is found to be given by $x = y = 1$ by setting these both to equal 0. We have (writing f_{xx} for $\partial^2 f/\partial x^2$ and so on), $f_{xx} = 2$, $f_{xy} = -2$ and $f_{yy} = 4$, so, since $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f(x, y)$ has a local minimum at $(1, 1)$.

5. The Lagrangean is $L = xy^{3/2} - \lambda(x + 2y - 100)$ and the first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x} = y^{3/2} - \lambda &= 0 \\ \frac{\partial L}{\partial y} = \frac{3}{2}xy^{1/2} - 2\lambda &= 0 \\ x + 2y &= 100. \end{aligned}$$

From the first two equations,

$$\lambda = y^{3/2} = \frac{3}{4}xy^{1/2},$$

which on simplification becomes $y = (3/4)x$. Then the third equation shows that $x = 40, y = 30$. There is no need to do a second-order test for the nature of the point: that is not part of the syllabus and there is no credit for it.

6. The system expressed in matrix form (a step some forgot) is

$$\begin{pmatrix} 4 & 1 & -2 \\ 2 & 3 & -2 \\ 2 & 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}.$$

(The system 'expressed in matrix form' is not the augmented matrix.) The standard matrix method approach is now to reduce the augmented matrix to reduced form. Here is one way. (There are others, equally valid. But note here how we have in several places been able to avoid introducing fractions by multiplying rows by appropriate constants before cancelling entries.)

$$\begin{pmatrix} 4 & 1 & -2 & 4 \\ 2 & 3 & -2 & 4 \\ 2 & 5 & 2 & 8 \end{pmatrix} \to \begin{pmatrix} 4 & 1 & -2 & 4 \\ 4 & 6 & -4 & 8 \\ 4 & 10 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 4 & 1 & -2 & 4 \\ 0 & 5 & -2 & 4 \\ 0 & 9 & 6 & 12 \end{pmatrix}$$

$$\to \begin{pmatrix} 4 & 1 & -2 & 4 \\ 0 & 45 & -18 & 36 \\ 0 & 45 & 30 & 60 \end{pmatrix} \to \begin{pmatrix} 4 & 1 & -2 & 4 \\ 0 & 45 & -18 & 36 \\ 0 & 0 & 48 & 24 \end{pmatrix} \to \begin{pmatrix} 4 & 1 & -2 & 4 \\ 0 & 5 & -2 & 4 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

It follows, from this last matrix, that

$$\begin{aligned} 4x + y - 2z &= 4 \\ 5y - 2z &= 4 \\ 2z &= 1. \end{aligned}$$

So, $z = 1/2$, $y = (1/5)(4+2z) = 1$ and $x = (1/4)(4+2z-y) = 1$. Simply manipulating equations does not constitute a matrix method, which the question specifically asks for, and gets no credit. Some candidates used Cramer's rule. This is a technique not discussed in the Subject Guide, and the computations required are more difficult. We recommend the reduction technique, but Cramer's rule, carefully applied, gives the correct answer.

7. For some a and r, the nth term of the sequence takes the form ar^{n-1}. The sum to infinity is $a/(1-r)$. So we must have

$$ar = 2, \quad \frac{a}{1-r} = 9.$$

These can now be solved simultaneously either by eliminating r or by eliminating a. Since the question asks for r, it is easiest to eliminate a. We see that, since $a = 9(1-r)$, we have $9(1-r)r = 2$, so $9r^2 - 9r + 2 = 0$, with solutions $r = 2/3, 1/3$. (Of course, alternatively, r may be eliminated, a found, and then r determined from a.)

8. (a) We have $TC = VC + FC = q(AVC) + FC = q^3 + 2q^2 + \ln(q^2+1) + 9$ and

$$MC = \frac{d}{dq}TC = 3q^2 + 4q + \frac{2q}{q^2+1}.$$

(b) The first integral is most easily done using a substitution. If we set $u = x+3$ the integral becomes $\int (u-3)^2 u^{1/2}\, du$, which is

$$\int u^{5/2} - 6u^{3/2} + 9u^{1/2}\, du = \frac{2}{7}u^{7/2} - \frac{12}{5}u^{5/2} + 6u^{3/2} + c = \frac{2}{7}(x+3)^{7/2} - \frac{12}{5}(x+3)^{5/2} + 6(x+3)^{3/2} + c.$$

The substitution $u = \sqrt{x+3}$ also works. So does integration by parts, though this is harder. (Some candidates attempted to use integration by parts, but only succeeded in ending up with a more complicated integral. No credit was given for this. Integration by parts is only useful if the resulting integral is easier than the one we start with.) For the second integral, we can use integration by parts:

$$\int x^{-2} \ln x\, dx = -\frac{1}{x}\ln x + \int \frac{1}{x}\frac{1}{x}\, dx = -\frac{1}{x}\ln x - \frac{1}{x} + c.$$

(c) This was not well done. Although the partial derivative with respect to x can be computed immediately as

$$\frac{\partial f}{\partial x} = -yx^{-y-1},$$

the derivative with respect to y is more difficult. For this, it is useful either to realise that $f = \exp(-y \ln x)$ or to take logarithms and differentiate. We have

$$\ln f = \ln(x^{-y}) = -y \ln x$$

and so, differentiating each side with respect to y,

$$\frac{1}{f}\frac{\partial f}{\partial y} = -\ln x,$$

so

$$\frac{\partial f}{\partial y} = -(\ln x) f = -(\ln x) x^{-y}.$$

9. (a) The total cost is $TC = q(AC) = 40 + 20q - 3q^2 + q^3/2$. Since the firm is a monopoly, we can find the price in terms of q from the demand equation. Hence the total revenue is $TR = pq = (40 - (5/2)q)q = 40q - (5/2)q^2$ and profit is

$$\Pi(q) = TR - TC = 20q + q - \frac{3}{2}q^2.$$

To maximise profit, we solve $\Pi'(q) = 0$, which means $3q^2 - 2q - 40 = 0$. This has the solutions $q = -3/10$ and $q = 4$, and it is the positive one that has economic meaning. We can see also that $\Pi''(4) < 0$, so $q = 4$ maximises profit.

(b) This is a constrained optimisation problem. We want to find the minimum cost of producing an amount q and, given the information in the question, this means we want to minimise $k + 16l$ subject to the constraint $k^{1/4}l^{1/4} = q$. The Lagrange multiplier method can then be used, with

$$L = k + 16l - \lambda \left(k^{1/4}l^{1/4} - q\right).$$

The equations to be solved are

$$\begin{aligned} \frac{\partial L}{\partial k} = 1 - \frac{1}{4}\lambda k^{-3/4}l^{1/4} &= 0 \\ \frac{\partial L}{\partial l} = 16 - \frac{1}{4}\lambda k^{1/4}l^{-3/4} &= 0 \\ k^{1/4}l^{1/4} &= q. \end{aligned}$$

From the first two equations, $\lambda = 4k^{3/4}l^{-1/4} = 64k^{-1/4}l^{3/4}$, so $k = 16l$ and then the third tells us that $(16)l^{1/4}l^{1/4} = q$, so $l^{1/2} = q/2$ and hence $l = q^2/4$, $k = 16l = 4q^2$ and the minimum cost is $4q^2 + 16(q^2/4) = 8q^2$.

10. (a) Carefully equating the supplies and demands, we obtain the following system of linear equations:

$$\begin{aligned} 4p_1 - 2p_2 + 4p_3 &= 50 \\ -2p_1 + 3p_2 - 4p_3 &= 20 \\ p_1 - 2p_2 + 4p_3 &= 35. \end{aligned}$$

Solving these using elementary row operations (but *not* simply by manipulating the equations, since the question explicitly asks that a matrix method be used) we find $p_1 = 5$, $p_2 = 60$ and $p_3 = 75/2$.

(b) We have $TC = \int MC\, dq = q + 2e^{0.5q} + \frac{1}{3}q^3 + c$ and, given that $TC(0) = FC = 10$, we have $0 + 2 + 0 + c = 10$, so $c = 8$ (and *not* 10!).

11. (a) We have total revenue given, as a function of q_1, q_2, by

$$TR = p_1q_1 + p_2q_2 = (1050 - 25q_1)q_1 + (500 - 2q_2)q_2,$$

and the profit is

$$\Pi(q_1, q_2) = TR - TC = 1000q_1 - 25q_1^2 + 480q_2 - q_2^2 - 10.$$

Solving $\partial\Pi/\partial q_1 = 0$ and $\partial\Pi/\partial q_2 = 0$ gives $q_1 = 20$ and $q_2 = 120$. The second order conditions are easily checked to show that these quantities do indeed maximise profit.

(b) Let y_N be the required number. Then we have

$$y_0 = 1000,$$

$$y_1 = (0.98)1000 + 30,$$

$$y_2 = (0.98)\left((0.98)1000 + 30\right) + 30 = (0.98)^2 1000 + (0.98)30 + 30,$$

$$y_3 = (0.98)\Big((0.98)^2 1000 + (0.98)30 + 30\Big) + 30 = (0.98)^3 1000 + (0.98)^2 30 + (0.98)30 + 30,$$

and, in general,

$$\begin{aligned} y_N &= (0.98)^N 1000 + (0.98)^{N-1}30 + (0.98)^{N-2}30 + \cdots + (0.98)30 + 30 \\ &= (0.98)^N 1000 + \Big(30 + 30(0.98) + 30(0.98)^2 + \cdots + 30(0.98)^{N-1}\Big) \\ &= (0.98)^N 1000 + \frac{30(1 - (0.98)^N)}{1 - 0.98} \\ &= 1000(0.98)^N + 1500(1 - (0.98)^N) \\ &= 1500 - 500(0.98)^N. \end{aligned}$$

This increases with N, converging to 1500.

Examination paper for 2005

The format of the 2005 examination will be the same as that of the 2004 examination; namely seven compulsory questions in Section A and two questions to be chosen from four in Section B.