Examination papers and Examiners' reports

Mathematics 1

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Undergraduate study in
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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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Examiner's report 2005

Zone A

Exam technique: general remarks

We start by emphasising that candidates should always include their working. This means two things. First, they should not simply write down the answer in the exam script, but explain the method by which it is obtained. Secondly, they should include all their working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing and that you have worked through all the steps to get to an answer: that is what is really being examined.

We also stress that if a student has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable on questions for these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, any topic could potentially appear in Section A.

Students are reminded that calculators are not permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.

Specific comments on questions

1. We have $2x^2 - x - 3 = 0 \iff (2x-3)(x+1) = 0$, so this curve crosses the x-axis when $x = -1$ and $x = 3/2$. It crosses the y-axis at $(0, -3)$. Furthermore, it has a U-shape. Similarly, $1 + x - 2x^2 = 0 \iff 2x^2 - x - 1 = 0 \iff (2x+1)(x-1) = 0$, so this crosses the x-axis when $x = 1$ and $x = -1/2$. It crosses the y-axis at $(0, 1)$ and it has a down-turned U-shape.

The curves intersect when $2x^2 - x - 3 = 1 + x - 2x^2$. This simplifies to $4x^2 - 2x - 4 = 0$ or $2x^2 - x - 2 = 0$, with solutions $x = (1 \pm \sqrt{17})/4$ (as can be seen by using the formula for the solutions of a quadratic equation).

There is no need to compute the y coordinate of the points where the curves intersect, since the question only asks for the x coordinates. Additionally, since quadratics always have their maximum or minimum halfway between their zeroes, there is no need to determine the stationary points using differentiation.

2. We have

$$f' = \frac{1}{2\sqrt{x}}e^{\sqrt{x}} - \frac{1}{\sqrt{x}}.$$

We solve $f' = 0$, which is

$$\frac{1}{2\sqrt{x}}(e^{\sqrt{x}} - 2) = 0,$$

so $\sqrt{x} = \ln 2$ and $x = (\ln 2)^2$. Note that

$$f'' = -\frac{1}{4}x^{-3/2}e^{\sqrt{x}} + \frac{1}{4}x^{-1}e^{\sqrt{x}} + \frac{1}{2}x^{-3/2}.$$

So,

$$f''((\ln 2)^2) = -\frac{1}{4}(\ln 2)^{-3}2 + \frac{1}{4}(\ln 2)^{-1}2 + \frac{1}{2}(\ln 2)^{-3} = \frac{1}{2}(\ln 2)^{-1} > 0.$$

It follows that the critical point is a minimum. Now, the corresponding value of f is $f((\ln 2)^2) = e^{\ln 2} - 2\ln 2 = 2 - 2\ln 2$.

3. We make the substitution $u = e^x$. We have $du = e^x\,dx$, so

$$I = \int \frac{1}{u}\frac{1}{1+u}\,du.$$

By partial fractions,

$$\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1} == \ln u - \ln(u+1) + c = \ln e^x - \ln(1+e^x) + c.$$

This can also be written as $x - \ln(1+e^x) + c$.

Alternatively, we could note that

$$I = \int \left(1 - \frac{e^x}{1+e^x}\right)\,dx,$$

from which the same results follows.

4. The augmented matrix is

$$\begin{pmatrix} -1 & 2 & 3 & 3 \\ 2 & -1 & 2 & 5 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

Using row operations to reduce, we have

$$\begin{pmatrix} -1 & 2 & 3 & 3 \\ 2 & -1 & 2 & 5 \\ 1 & 1 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & -1 & 2 & 5 \\ -1 & 2 & 3 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & 3 & 4 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \end{pmatrix},$$

so we have $z = 1$, $y = 1$ and $x = 4 - y - z = 2$.

Be careful that in answering a question like this, you use only valid row operations. For example, multiplying two rows together or subtracting a fixed number from each entry of a row, is not permissible.

Methods such as Cramer's rule can be successfully used to solve this question (although these techniques are not part of the formal Mathematics 1 syllabus).

5. The partials are

$$f_x = 4x^3 + 4xy$$
$$f_y = 2x^2 + 4y + 2.$$

We solve $f_x = f_y = 0$. Now, $f_x = 0$ means $x(x^2+y) = 0$, so $x = 0$ or $y = -x^2$. Suppose $x = 0$. Then, from $f_y = 0$ we have $4y + 2 = 0$, so $y = -1/2$. Now suppose $y = -x^2$. From $f_y = 0$ we have $2x^2 - 4x^2 + 2 = 0$, which means $x = \pm 1$.

So the critical points are $(0, -1/2)$, $(1, -1)$ and $(-1, -1)$.

The second derivatives are

$$f_{xx} = 12x^2 + 4y, \quad f_{yy} = 4, \quad f_{xy} = 4x.$$

At $(0, -1/2)$ we have $f_{xx}f_{yy} - f_{xy}^2 < 0$, so this is a saddle point.

At $(1, -1)$ we have $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ so this is a local minimum.

At $(-1, -1)$, $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ so this is a local minimum.

6. The Lagrangean is $L = x + 2y - \lambda(x^3y^2 - 27)$. The first order conditions are

$$1 - 3\lambda x^2y^2 = 0,$$
$$2 - 2\lambda x^3 y = 0,$$
$$x^3y^2 = 27.$$

The first two imply that $\dfrac{1}{3x^2y^2} = \dfrac{1}{x^3y}$, and so $x = 3y$. Then, $(3y)^3y^2 = 27$, which is $y^5 = 1$, so

$$x = 3, y = 1.$$

There is no need to check that these values do indeed give a minimum: that method is not part of this syllabus, and there is no credit for applying it, so it is not something to spend time on. (It is not as straightforward as some candidates seemed to think. The second order test for constrained problems does not simply involve the second derivatives of L with respect to x and y.)

7. Let y_N be the balance after N years. (Make it clear what you are trying to determine.) So $y_0 = A$, $y_1 = (1.05)A - W$,

$$y_2 = (1.05)\left((1.05)A - W\right) - W = (1.05)^2 A - (1.05)W - W,$$

$$y_3 = (1.05)\Big((1.05)^2 A - (1.05)W - W\Big) - W = (1.05)^3 A - (1.05)^2 W - (1.05)W - W,$$

and, in general

$$y_N = (1.05)^N A - (1.05)^{N-1} W - (1.05)^{N-2} W - \cdots - W.$$

This simplifies (noting the geometric progression) to

$$y_N = (1.05)^N A - W\frac{1-(1.05)^N}{1-1.05} = (1.05)^N A + 20W(1-(1.05)^N).$$

Some candidates used recurrence/difference equations (a technique not formally part of Mathematics 1), and this is perfectly acceptable.

8.(a) Since $AVC = (TC - FC)/q$, the total cost is $TC = q(AVC) + FC$, which is $q^5 - q^2 - 5q^3 + 5$. The total revenue is $TR = pq = (20-q)q$, so the profit is $\Pi = TR - TC = 20q - q^5 + 5q^3 - 5$. We solve $\Pi' = 0$. This is $20 - 5q^4 + 15q^2 = 0$, which we note is a quadratic in q^2, which simplifies as $-5(q^2-4)(q^2+1) = 0$. This means $q^2 = 4$, to which the only economically meaningful solution is $q = 2$. (We could equally well use the formula for the solutions to a quadratic to see that $q^2 = 4$.) Furthermore, since $\Pi'' = -20q^3 + 30q$, we have $\Pi''(2) < 0$ and therefore we maximise profit.

(b) Substituting $u = x + 1$ in the first integral, we have $du = dx$ and so the integral is

$$\int \frac{u-1}{u^3}\,du = \int (u^{-2} - u^{-3})\,du = -u^{-1} + u^{-2}/2 + c = -\frac{1}{x+1} + \frac{1}{2(x+1)^2} + c.$$

Alternatively, we can use integration by parts. This gives:

$$\int \frac{x}{(1+x)^3}\,dx = \int x(x+1)^{-3}\,dx$$

$$= -\frac{1}{2}(x+1)^{-2}x + \int \frac{1}{2}(x+1)^{-2} = -\frac{1}{2}(x+1)^{-2}x - \frac{1}{2}(x+1)^{-1} + c.$$

Next, for the second integral, let $u = \ln x$. We have $du = (1/x)\,dx$ so the integral is

$$\int \frac{u}{u^2+1}\,du.$$

Now let $v = u^2 + 1$ and the integral becomes

$$\int \frac{dv}{2v}\,dv = \frac{1}{2}\ln v + c = \frac{1}{2}\ln((\ln x)^2 + 1) + c.$$

9.(a) From $f(1) = 5/4$ we obtain $a + b + c/4 = 5/4$. Next, we have

$$f'(x) = 2ax - \frac{2c}{(1+x)^3},$$

so $f'(1) = 15/4$ means $2a - c/4 = 15/4$. Next,

$$\int_0^1 f(x)\,dx = \left[\frac{ax^3}{3} + bx - \frac{c}{1+x}\right]_0^1 = \frac{a}{3} + b + \frac{c}{2},$$

and so we have $a/3 + b + c/2 = 1/6$.

Multiplying these three equations appropriately gives the required system.

Reducing the augmented matrix,

$$\begin{pmatrix} 4 & 4 & 1 & 5 \\ 8 & 0 & -1 & 15 \\ 2 & 6 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 6 & 3 & 1 \\ 4 & 4 & 1 & 5 \\ 8 & 0 & -1 & 15 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 6 & 3 & 1 \\ 0 & -8 & -5 & 3 \\ 0 & -24 & -13 & 11 \end{pmatrix} \to \begin{pmatrix} 2 & 6 & 3 & 1 \\ 0 & 8 & 5 & -3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

so $c = 1$, $b = (-3 - 5c)/8 = -1$ and $a = (1 - 3c - 6b)/2 = 2$.

(b) We have

$$\frac{\partial f}{\partial x} = x\ln(x/y) + xy(y/x)(1/y) = y\ln(x/y) + y$$

and

$$\frac{\partial f}{\partial y} = x\ln(x/y) + xy(y/x)(-x/y^2) = x\ln(x/y) - x.$$

10.(a) We use the Lagrange multiplier method. We have

$$L = (x+2)^2(y+1)^3 - \lambda(px + qy - M)$$

and the first order conditions are

$$2(x+2)(y+1)^3 - p\lambda = 0,$$

$$3(x+2)^2(y+1)^2 - q\lambda = 0,$$

$$px + qy = M.$$

The first two equations imply $2(y+1)/p = 3(x+2)/q$, so that

$$y = \frac{3p}{2q}x + \frac{3p}{q} - 1.$$

Now, the third equation tells us

$$px + \frac{3p}{2}x + 3p - q = M = M,$$

so

$$x = \frac{2M}{5p} + \frac{2q}{5p} - \frac{6}{5}$$

and then

$$y = \frac{3p}{2q}x + \frac{3p}{q} - 1 = \frac{3M}{5q} + \frac{6p}{5q} - \frac{2}{5}.$$

(b) We have $p_X = 76 - x$ and $p_Y = 100 - 2y$, so that the profit function is

$$\Pi = xp_X + yp_Y - TC = x(76 - x) + y(100 - 2y) - (2x^2 + 2xy + 3y^2 + 10)$$
$$= 76x - 3x^2 - 5y^2 + 100y - 2xy - 10.$$

We solve $\Pi_x = \Pi_y = 0$, which is $76 - 6x - 2y = 0, \quad -10y + 100 - 2x = 0$. This is equivalent to $3x + y = 38, \quad x + 5y = 50$, which has solution $x = 10, y = 8$.

We note that $\Pi_{xx} = -6 < 0$ and $\Pi_{xx}\Pi_{yy} - \Pi_{xy}^2 = (-6)(-10) - (-2)^2 > 0$, so it is a maximum.

11.(a) It's important to realise at the outset that the problem is to minimise $xc_X + yc_Y$ subject to $\sqrt{x} + \sqrt{y} = q$. The Lagrangian is $L = xc_X + yc_Y - \lambda(\sqrt{x} + \sqrt{y} - q)$. The first order conditions are

$$c_X - \frac{1}{2\sqrt{x}}\lambda = 0,$$
$$c_Y = \frac{1}{2\sqrt{y}}\lambda = 0,$$
$$\sqrt{x} + \sqrt{y} = q.$$

The first two equations give $\sqrt{x}c_X = \sqrt{y}c_Y$, from which we have $y = (c_X^2/c_Y^2)x$. Then the third equation becomes $\sqrt{x} + (c_X/c_Y)\sqrt{x} = q$ and hence $x = \left(\frac{c_Y}{c_X + c_Y}\right)^2 q^2$ and

$$y = \frac{c_X^2}{c_Y^2}\frac{c_Y^2}{(c_X + c_Y)^2}q^2 = \frac{c_X^2}{(c_X + c_Y)^2}q^2.$$

The minimised cost is therefore

$$C = xc_X + yc_Y = \left(\frac{c_X c_Y^2}{(c_X + c_Y)^2} + \frac{c_Y c_X^2}{(c_X + c_Y)^2}\right) q^2 = \frac{c_X c_Y (c_X + c_Y)}{(c_X + c_Y)^2} = \frac{c_X c_Y q^2}{(c_X + c_Y)}.$$

The corresponding λ is

$$\lambda^* = 2\sqrt{x}c_X = 2c_X \frac{c_Y}{c_X + c_Y} q = \frac{2c_X c_Y}{(c_X + c_Y)} q$$

and

$$\frac{\partial C}{\partial q} = 2\frac{c_X c_Y}{(c_X + c_Y)} q = \lambda^*$$

(b) Let the first term be a and the common ratio be r. Then the nth term is ar^{n-1} and the sum of the first five terms is $\frac{a(1-r^5)}{1-r}$. We therefore have $a(1-r^5)/(1-r) = 31$ and $ar^5 = 16ar$. So, $r^4 = 16$ and hence $r = 2$ or -2. Then, $a = 31(1-r)/(1-r^5) = 1$ or $31/11$.

Examination paper for 2006

There will be no change to the format, style or number of questions in the examination paper for 2006.