Examination papers and Examiners' reports

Mathematics 2

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Undergraduate study in
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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

UNIVERSITY OF LONDON EXTERNAL PROGRAMME 1858

Examiner's report 2005

Zone A

Exam technique: general remarks

We start by emphasising that candidates should always include their working. This means two things. First, they should not simply write down the answer in the exam script, but explain the method by which it is obtained. Secondly, they should include all their working. The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing and that you have worked through all the steps to get to an answer: that is what is really being examined.

We also stress that if a student has not completely solved a problem, they may still be awarded marks for a partial, incomplete, or slightly wrong solution; but, if they have written down a wrong answer and nothing else, no marks can be awarded.

Candidates should ensure that they have covered the bulk of the course in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable on questions for these topics. There are no formal options in this course: all students should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, any topic could potentially appear in Section A.

Students are reminded that calculators are not permitted in the examination for this subject, under any circumstances. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this subject.

Specific comments on questions

1. To find the equilibrium price, we solve $2400 - p = 2p$, which gives $p = 800$. The equilibrium quantity is $q = 2(800) = 1600$. With a tax of 25%, the demand equation stays the same but the supply equation becomes $q = 2p(1 - 0.25)$. Make sure you understand this: see the Subject Guide. It is a common mistake to change the demand equation rather than the supply equation. We now solve

$$2400 - p = 2p(1 - 0.25),$$

which gives a new equilibrium price of $p = 960$. The corresponding quantity is either $2400 - 960$ or $2(960)(1 - 0.25)$. (It is *not* $2(960)$.) This gives $q = 1440$.

When there is an excise tax of T per unit, the supply equation is $q = 2(p - T)$ and the demand equation, unchanged, is $2400 - p$. Now, we could solve $2400 - p = 2(p - T)$ to obtain an expression for T in terms of p, and then set $p = 960$ to find T. But we can find T more quickly by noting that it must be the case that $T = 0.25p = 0.25(960) = 240$.

2.(a) There was a misprint in this question. It should not have asked for γ since there is no γ involved in the function. This was not the kind of error that would result in candidates wasting any time and, indeed, many quite rightly noted that there was no γ. Generally, please be assured that whenever a question has a misprint or error, the examiners make full allowance for it. Since the function is homogeneous, the numerator must be homogeneous, and so $\alpha - 1 = 2 + \beta$. Since the degree of homogeneity is D and since the denominator has degree 2, we also have $(\alpha - 1) - 2 = D$. Solving these equations, we see that $\alpha = D + 3$ and $\beta = D$.

(b) First check that the function is homogeneous by noting that $f(cx, cy) = cf(x, y)$, which also shows the degree of homogeneity to be 1. Euler's equation is therefore

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f.$$

The derivatives are

$$\frac{\partial f}{\partial x} = -\sin\left(\frac{x}{y}\right), \quad \frac{\partial f}{\partial y} = \cos\left(\frac{x}{y}\right) + \frac{x}{y}\sin\left(\frac{x}{y}\right).$$

Euler's equation is then easy to check.

3. For the equilibrium quantity, we solve

$$\frac{210}{q^2 + 5q + 6} = 5,$$

which simplifies to $q^2 + 5q + 6 = 42$ and hence $q^2 + 5q - 36 = 0$. This factorises as $(q + 9)(q - 4)$ and hence $q = 4$ is the only economically meaningful solution. To find the consumer surplus, we calculate

$$\int_0^4 \frac{210}{q^2 + 5q + 6}\, dq - 5(4).$$

The integral can be determined using partial fractions and the answer is

$$210(\ln 6 - \ln 7 - \ln 2 + \ln 3) - 20 = 210 \ln\left(\frac{9}{7}\right) - 20.$$

The next part of the question was a bit ambiguous. It should have been interpreted as asking what the demand function must be if, *instead* of being the function given in the question, it was one which had constant elasticity of 2, and with the same equilibrium. The examiners were very generous with other interpretations of the question. To find the demand function in this case, we solve

$$-\frac{p}{q}\frac{dq}{dp} = 2.$$

This is a separable differential equation. We separate and integrate to see that

$$\int \frac{dq}{q} = -\int 2\frac{dp}{p},$$

so

$$\ln q = -2\ln p + c$$

and hence $q = A/p^2$, (where $A = e^c$). Using the fact that $q(5) = 4$, we find $A = 100$, so $q = 100/p^2$. A common error was to forget the $+c$.

4. Reducing the augmented matrix, we have

$$\begin{pmatrix} 2 & -1 & 1 & 4 \\ -1 & 3 & 2 & 3 \\ 1 & 2 & 3 & 7 \\ 1 & -2 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 7 \\ 2 & -1 & 1 & 4 \\ -1 & 3 & 2 & 3 \\ 1 & -2 & -1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & 5 & 5 & 10 \\ 0 & -4 & -4 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(Of course, other reductions are possible.) Now, z can take any value, so there are infinitely many solutions. If we set $z = s$ then it follows that $y = 2 - s$ and $x = 7 - 2y - 3z = 3 - s$. This gives a solution for each real number s.

5. This is *not* a first-order difference equation. It is in fact a second-order equation, because it is $x_{t+1} = ax_{t-1}$, rather than $x_{t+1} = ax_t$. In standard form, the equation is $x_{t+1} - ax_{t-1} = 0$, so the auxiliary equation is $z^2 - a = 0$, with solutions $\sqrt{a}$ and $-\sqrt{a}$. Therefore, for some constants A and B,

$$x_t = A(\sqrt{a})^t + B(-\sqrt{a})^t.$$

The facts that $x_0 = 1$ and $x_1 = 3\sqrt{a}$ show that $A = 2$ and $B = -1$, so

$$x_t = 2(\sqrt{a})^t - (-\sqrt{a})^t.$$

6. We have $\dfrac{dp}{dt} = (8 - 4p)^2$. Separating and integrating,

$$\int \frac{dp}{(8-4p)^2} = \int dt = t + c,$$

so

$$\frac{1}{4}\frac{1}{8-4p} = t + c.$$

Now, $p(0) = 1$ implies that $c = 1/16$. Solving for p, we obtain

$$p(t) = 2 - \frac{1}{16t+1}$$

and we have $p(t) \to 2$ as $t \to \infty$.

7.(a) We first find the eigenvalues and corresponding eigenvectors. The eigenvalues are 3 and 6 and corresponding eigenvectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Setting these as the columns of a matrix P, a suitable P and D are

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$$

There is no need to compute $P^{-1}AP$ explicitly to determine D: you can simply state the result because of the underlying theory. There are various ways we can now proceed to solve the system of differential equations. As discussed in the Subject Guide, we can use diagonalisation, or we can reduce the system to a single second-order differential equation. With the diagonalisation approach, we introduce two new functions F, G defined by

$$\begin{pmatrix} f \\ g \end{pmatrix} = P \begin{pmatrix} F \\ G \end{pmatrix}.$$

The functions F, G then satisfy

$$\begin{pmatrix} dF/dx \\ dG/dx \end{pmatrix} = D \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 3F \\ 6G \end{pmatrix},$$

so that, for some constants A and B, $F = Ae^{3x}$ and $G = Be^{6x}$. Then,

$$\begin{pmatrix} f \\ g \end{pmatrix} = P \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} Ae^{3x} \\ Be^{6x} \end{pmatrix} = \begin{pmatrix} Ae^{3x} + Be^{6x} \\ -2Ae^{3x} + Be^{6x} \end{pmatrix}.$$

The facts that $f(0) = 2$ and $g(0) = 1$ mean that $A = 1/3$ and $B = 5/3$. So,

$$f(x) = \frac{1}{3}e^{3x} + \frac{5}{3}e^{6x}, \quad g(x) = -\frac{2}{3}e^{3x} + \frac{5}{3}e^{6x}.$$

(b) We have

$$\cos y \simeq 1 - \frac{y^2}{2} + \frac{y^4}{24}$$

and

$$\ln(1 - x) \simeq -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}.$$

Hence,

$$\cos(\ln(1-x)) \simeq 1 - \frac{1}{2}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)^2 + \frac{1}{24}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)^4.$$

We only want powers of x up to x^4, so the only term we need from the last of the terms above is $x^4/24$. Carefully determining the relevant terms from the squared term, and collecting terms together, we should obtain

$$\cos(\ln(1-x)) \simeq 1 - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{5}{12}x^4.$$

8.(a) We have

$$Y_t = \frac{1}{8}Y_{t-1} + 100 + \frac{1}{32}(Y_{t-1} - Y_{t-2}) + \frac{3}{32}Y_{t-1}.$$

Simplifying this and expressing it in standard form, we have the second-order difference equation

$$Y_t - \frac{1}{4}Y_{t-1} + \frac{1}{32}Y_{t-2} = 100,$$

with auxiliary equation $z^2 - z/4 + 1/32 = 0$. This has no real roots. A particular solution is easily seen to be $Y^* = 128$. Applying the method described in the Subject Guide, the solution to the equation is therefore, for some constants A and B,

$$Y_t = 128 + \left(\frac{\sqrt{2}}{8}\right)^t \left(A\sin\left(\frac{\pi t}{4}\right) + B\cos\left(\frac{\pi t}{4}\right)\right).$$

The facts that $Y_0 = 129$ and $Y_1 = 128.25$ enable us to determine that $A = B = 1$ and hence the solution is

$$Y_t = 128 + \left(\frac{\sqrt{2}}{8}\right)^t \left(\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{4}\right)\right).$$

(b) Reducing the augmented matrix, we have

$$\begin{pmatrix} 1 & 1 & -3 & 4 \\ 2 & -1 & 1 & 3 \\ 1 & 4 & a & b \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 & 4 \\ 0 & -3 & 7 & -5 \\ 0 & 3 & a+3 & b-4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 & 4 \\ 0 & -3 & 7 & -5 \\ 0 & 0 & a+10 & b-9 \end{pmatrix}.$$

This will be consistent with infinitely many solutions if and only if the last row is all-zeros, which means $a = -10$ and $b = 9$. It will have exactly one solution if and only if $a + 10 \neq 0$, so $a \neq -10$. It will be inconsistent (no solutions) if and only if $a + 10 = 0$ and $b - 9 \neq 0$, that is $a = -10$ and $b \neq 9$.

9.(a) The inverse can be determined either by using determinant-based techniques, or by using row operations. The answer is

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

(b) In standard form, the differential equation is

$$\frac{d^2f}{dx^2} - \frac{df}{dx} - (a + a^2)f = 1 + a^2x + ax.$$

The auxiliary equation, $z^2 - z - (a + a^2) = 0$ has solutions

$$z = \frac{1}{2}\left(1 \pm \sqrt{1 + 4a + 4a^2}\right) = \frac{1}{2}\left(1 \pm \sqrt{(1+2a)^2}\right) = \frac{1}{2}\left(1 \pm (1 + 2a)\right),$$

so the solutions are $a+1$ and $-a$. For a particular solution, we try $y = c + dx$. Substitution into the equation shows that we need $d = -1$ and $c = 0$. So, for some constants A and B,

$$f(x) = Ae^{(a+1)x} + Be^{-ax} - x.$$

Noting that $f'(x) = (a+1)A^{(a+1)x} - aBe^{-ax} - 1$ and using the facts that $f(0) = 0$ and $f'(0) = -2a - 1$, we have

$$A + B = 0, \quad (a+1)A - aB - 1 = -2a - 1,$$

with solutions $A = -2a/(2a+1)$ and $B = 2a/(2a+1)$. So, finally,

$$f(x) = \frac{2a}{2a+1}\left(e^{-ax} - e^{(a+1)x}\right) - x.$$

10.(a) The Lagrangian is

$$L = x^{-1}y^{-2}z^{-3} - \lambda(px + qy + rz - c)$$

and the first order conditions are

$$-x^{-2}y^{-2}z^{-3} - \lambda p = 0,$$
$$-2x^{-1}y^{-3}z^{-3} - \lambda q = 0,$$
$$-3x^{-1}y^{-2}z^{-4} - \lambda r = 0,$$
$$px + qy + rz = c.$$

Eliminating λ, it follows that

$$\frac{x^{-2}y^{-2}z^{-3}}{p} = \frac{2x^{-1}y^{-3}z^{-3}}{q} = \frac{3x^{-1}y^{-2}z^{-4}}{r}.$$

This means that $y = (2p/q)x$ and $z = (3p/r)x$. Then, $px + qy + rz = c$ shows that $x = c/(6p)$. It follows that $y = c/(3q)$ and $z = c/(2r)$. The minimum value is therefore $c^{-6}(6p)(3q)^2(2r)^3 = 432pq^2r^3/c^6$.

(b) The supply function is $q^S(p) = -4 + (1/3)p$, the inverse supply function is $p^S(q) = 12 + 3q$, the demand function is $q^D(p) = 20 - p$, and the inverse demand function is $p^D(q) = 20 - q$. Now, we have

$$p_t = p^D(q_t) = 20 - q_t = 20 - q^S(p_{t-1}) = 20 - (-4 + p_{t-1}/3),$$

so

$$p_t = -\frac{1}{3}p_{t-1} + 24.$$

This is a first-order difference equation. The time-independent solution is

$$p^* = \frac{24}{1-(-1/3)} = 18$$

and the solution is

$$p_t = 18 + (10 - 18)(-1/3)^t = 18 - 8(-1/3)^t.$$

As $t \to \infty$, $p_t \to 18$. Now,

$$q_t = -4 + p_{t-1}/3 = 2 - \frac{8}{3}\left(-\frac{1}{3}\right)^{t-1} \to 2$$

as $t \to \infty$.

Examination paper for 2006

There will be no change to the format, style or number of questions in the examination paper for 2006.