05b Revision Problems – Solutions

1 One approach is to directly use Taylor's theorem, calculating $f'(0), f''(0)$ and $f^{(3)}(0)$, and using

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3.$$

However, an easier approach is to use the standard results (which you may assume) concerning the series for e^y and $(1-x)^{-1}$, which are

$$e^y \simeq 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots$$

and

$$(1-x)^{-1} \simeq 1 + x + x^2 + x^3 + \cdots.$$

Then,

$$e^{2x}(1-x)^{-1} \simeq \left(1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{6}\right)\left(1 + x + x^2 + x^3\right),$$

which, after careful manipulation, and omitting powers of x higher than 3, becomes

$$1 + 3x + 5x^2 + \frac{19}{3}x^3.$$

2 We use the fact that, if the function is homogeneous of degree 2, then:

- The numerator must be homogeneous
- The denominator must be homogeneous
- The degree of the numerator must equal 2 more than the degree of the denominator

Respectively, these three conditions mean:

1. $\alpha + 1 = 2 + \beta$
2. $1 + \gamma = 4 + 2\delta$
3. We can write four possible equations here. For example, since the degree of the first term on the numerator is $\alpha + 1$ and the degree of the first term on the denominator is $(1+\gamma)/4$ (noting that the denominator consists of an expression raised to the power of $1/4$), we have $\alpha - 1 - (1+\gamma)/2 = 2$.

Solving these to find β, γ and δ in terms of α, we find that $\beta = \alpha - 1$, $\gamma = 4\alpha - 5$ and $\delta = 2\alpha - 4$.

3. (a) We solve $113-q^2 = 1+q^2+2q$, which, in standard form, is $q^2+q-56 = 0$, with positive solution $q = 7$, which is therefore the equilibrium quantity. The corresponding price is $p = 113 - q^2 = 64$. To find the consumer and producer surplus, it might be useful to sketch roughly the graphs so that we know what areas to determine. The consumer surplus is

$$CS = \int_0^7 (113 - q^2)\, dq - 7(64) = 686/3.$$

The producer surplus is

$$PS = 7(64) - \int_0^7 (1 + q^2 + 2q)\, dq = 833/3.$$

(b) This is a separable equation. Separating and integrating, we have

$$\int \frac{dy}{y^2} = -\int \frac{t}{\sqrt{1+t^2}}\, dt,$$

so

$$-\frac{1}{y} = -\sqrt{1+t^2} + c$$

and so

$$y = \frac{1}{\sqrt{1+t^2} - c}.$$

Since $y(0) = 2$, we must have $1/(1-c) = 2$, so $c = 1/2$ and

$$y = \frac{1}{\sqrt{1+t^2} - 1/2} = \frac{2}{2\sqrt{1+t^2} - 1}.$$

4 The equilibrium price is given by $5p - 2 = 12 - 2p$, so $p = 2$. The equilibrium quantity is 8. When a tax of the type described is imposed, the new equilibrium price is given by

$$5p(1-r) - 2 = 12 - 2p.$$

(Note that we change the supply equation but not the demand equation. The reasoning is that, as far as the suppliers are concerned the effective price is the price minus the tax, which is $p(1-r)$.) Solving for p, we obtain $p = 14/(7-5r)$. Then $q = 12 - 2p = (56-60r)/(7-5r)$. The tax revenue per unit is rp, so the total tax revenue is rpq, which is

$$r\frac{14}{(7-5r)}\frac{(56-60r)}{(7-5r)} = \frac{14r(56-60r)}{(7-5r)^2}.$$

5 The elasticity is given by

$$\varepsilon = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{q}\frac{8-5p^2}{(p^2+8)^4},$$

where we've omitted the details of the differentiation. Now, this is not yet the answer, because we need to find the elasticity in terms of p only. We must therefore eliminate q:

$$\varepsilon = -\frac{p}{q}\frac{8-5p^2}{(p^2+8)^4} = -\frac{p}{p/(p^2+8)^3}\frac{8-5p^2}{(p^2+8)^4} = \frac{5p^2-8}{p^2+8}.$$

Demand is elastic if and only if $\varepsilon > 1$, which is equivalent to $5p^2 - 8 > p^2 + 8$, or $4p^2 > 16$. So (noting that $p \geq 0$) we need $p > 2$.

6) There are various approaches. One is to directly use Taylor's theorem, calculating $f'(0), f''(0), f^{(3)}(0)$, $f^{(4)}(0)$ and $f^{(5)}(0)$ and using

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5.$$

These successive derivatives do, however, get harder to compute. An easier approach is note that $f(x) = \ln(1+2x) - \ln(1+x)$ and to use the standard result (which you may assume) concerning the series for $\ln(1+y)$, which is

$$\ln(1+y) \simeq y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} + \cdots$$

Taking $y = 2x$ and x, we obtain

$$\begin{aligned}
f(x) &= \ln(1+2x) - \ln(1+x) \\
&\simeq \left(2x - \frac{1}{2}(2x)^2 + \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 + \frac{1}{5}(2x)^5\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) + \\
&= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots \\
&= x - \frac{3}{2}x^2 + \frac{7}{3}x^3 - \frac{15}{4}x^4 + \frac{31}{5}x^5 + \cdots.
\end{aligned}$$

7 The terms on the top must both have the same degree, so $\alpha+\beta+2=1+1=2$, so $\alpha+\beta=0$. Similarly, the terms on the bottom must have the same degree as each other, so $\gamma=2$. Since the function has degree 0, the terms on the top have the same degree as those on the bottom, so we can write $\alpha+\beta+2-2\delta=0$ (or, equally, $2-2\delta=0$). So, we have $\delta=1$. Summarising, what we have is $\alpha+\beta=0$, $\gamma=2$ and $\delta=1$.

8 (a) To find the equilibrium, we can equate the demand and supply prices, or the demand and supply quantities. Taking the former approach, we have $p=\dfrac{20}{q+3}=q+4$. This is equivalent to $(q+4)(q+3)=20$ which is $q^2+7q-8=0$. Taking the positive solution, we have $q=1$. Then $p=1+4=5$ (or, alternatively, $p=20/(1+3)=5$). To find the consumer surplus, it is useful to sketch roughly the graphs so that we know what are to determine. The solution is

$$CS=\int_0^1 \frac{20}{q+3}dq-(1\times 5)=[20\ln(q+3)]_0^1-5=20\ln 4-20\ln 3-5=20\ln(4/3)-5.$$

(b) The equation for p becomes

$$\frac{dp}{dt}=((1-2p)-2p)^3=(1-4p)^3.$$

This is separable, and we have:

$$\int \frac{dp}{(1-4p)^3}=\int dt,$$

so

$$\frac{1}{8}\frac{1}{(1-4p)^2}=t+c.$$

Given that $p(0)=1/8$, $c=1/2$. From

$$\frac{1}{(1-4p)^2}=8(t+1/2)=8t+4,$$

we have

$$(1-4p)=\pm\frac{1}{\sqrt{8t+4}}.$$

Given that $p(0)=1/8$, we take the $+$ sign and end up with

$$p=\frac{1}{4}-\frac{1}{4}\frac{1}{\sqrt{8t+4}},$$

which tends to $1/4$ as t tends to infinity.

9 (The system 'expressed in matrix form' is not the augmented matrix.) We can answer the question using reduction. One possible reduction is as follows.

$$\begin{pmatrix} 2 & 1 & -3 & 2 \\ 1 & -1 & 2 & 2 \\ 3 & 3 & c & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 2 \\ 2 & 1 & -3 & 2 \\ 3 & 3 & c & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & -7 & -2 \\ 0 & 6 & (c-6) & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & -7 & -2 \\ 0 & 0 & (c+8) & 0 \end{pmatrix}.$$

There will be infinitely many solutions if and only if this last row is an all-zero row, meaning that $c = -8$. In this case, the system is equivalent to the two equations

$$\begin{aligned} x - y + 2z &= 2 \\ 3y - 7z &= -2. \end{aligned}$$

Let $z = r$, any number. Then, from the second of these two equations, $y = -2/3 + (7/3)r$ and, then, from the first, $x = 4/3 + (r/3)$. (Another approach to determining the value of c is to calculate the determinant of the coefficient matrix, finding that it is $-3c - 24$. Now, by the theory of linear equations, if the determinant is non-zero then the system has exactly one solution. So it can only have infinitely many if $c = -8$. But then you cannot proceed and use Cramer's rule or matrix inversion to *show* that it does indeed have infinitely many solutions in this case and to *find* them all: the reduction method has to be used to complete the problem.)

(b) We solve $\partial f/\partial x = 0 = \partial f/\partial y$ simultaneously. This gives $16x + 8y + 12 = 0 = 8x + 20y + 10$, with solution $x = -5/8$ and $y = -1/4$. The second derivative test must then be applied to prove that the critical point is a minimum: this requires noting *both* that $\partial^2 f/\partial x^2 > 0$ *and*

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

10 (a) This is a constrained optimisation problem. We want to find the values of x_1 and x_2 that maximise the utility function $x_1 x_2^2$ subject to the budget constraint $p_1 x_1 + p_2 x_2 = M$. The Lagrange multiplier method can then be used, with

$$L = x_1 x_2^2 - \lambda \left(p_1 x_1 + p_2 x_2 - M\right).$$

The equations to be solved are

$$\begin{aligned} \frac{\partial L}{\partial x_1} = x_2^2 - \lambda p_1 &= 0 \\ \frac{\partial L}{\partial x_2} = 2x_1 x_2 - \lambda p_2 &= 0 \\ p_1 x_1 + p_2 x_2 &= M. \end{aligned}$$

From the first two equations,

$$\lambda = \frac{x_2^2}{p_1} = \frac{2x_1 x_2}{p_2},$$

and, since $x_2 = 0$ is clearly not going to maximise the utility function, we may assume $x_2 \neq 0$, and cancel to obtain $x_2 = 2(p_1/p_2)x_1$. Then the third equation shows that $p_1x_1 + 2p_1x_1 = M$, so $x_1^* = M/(3p_1)$ and $x_2^* = 2M/(3p_2)$. There is no need to prove that this maximises the utility: second-order tests for constrained optimisation are not included in this subject. The corresponding value of λ is

$$\lambda^* = \frac{(x_2^*)^2}{p_1} = \frac{4M^2}{9p_1p_2^2}.$$

For the last part, there is no need to calculate V: we can simply use the fact that $\partial V/\partial M = \lambda^*$ (and we have just calculated λ^*).

(b) We have

$$Y_t = C_t + I_t = 10 + \frac{7}{9}Y_{t-1} + 50 + \frac{2}{9}Y_{t-1} - \frac{2}{9}Y_{t-1},$$

so $Y_t - Y_{t-1} + (2/9)Y_{t-2} = 60$. A particular solution is given by $Y_t = k$, where $k - k + (2/9)k = 60$, so that $k = 270$. The auxiliary equation, $z^2 - z + 2/9 = 0$ has solutions $2/3, 1/3$, so

$$Y_t = A\left(\frac{2}{3}\right)^t + B\left(\frac{1}{3}\right)^t + 270,$$

for some constants A and B. $Y_0 = 271$ means $A + B = 1$ and $Y_1 = 270$ means $(2/3)A + (1/3)B = 0$, so $A = -1$ and $B = 2$. Therefore,

$$Y_t = 270 - \left(\frac{2}{3}\right)^t + 2\left(\frac{1}{3}\right)^t.$$

11 (a) This was an unusual question, but clear guidance, step by step, is given.

(i) We have

$$p_n - p_{n-1} = \frac{1}{4}(1 - 2p_{n-1}) + c(-1)^n,$$

so

$$p_n - \frac{1}{2}p_{n-1} = \frac{1}{4} + c(-1)^n.$$

(ii) If $c = 0$, then we need to solve $p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}$. This has constant solution $p^* = (1/4)/(1 - (1/2)) = 1/2$, so

$$p_n = p^* + (p_0 - p^*)\left(\frac{1}{2}\right)^n = \frac{1}{2} + A\left(\frac{1}{2}\right)^n,$$

where A is a constant.

(iii) As suggested, we try

$$p_n = \frac{1}{2} + A\left(\frac{1}{2}\right)^n + d(-1)^n.$$

Then,

$$p_n - \frac{1}{2}p_{n-1} = \left(\frac{1}{2} + A\left(\frac{1}{2}\right)^{n-1} + d(-1)^n\right) - \left(\frac{1}{4} + \frac{A}{2}\left(\frac{1}{2}\right)^n + \frac{d}{2}(-1)^{n-1}\right) = \frac{1}{4} + \frac{3d}{2}(-1)^n.$$

For this to equal $1/4 + 3(-1)^n$, we therefore take $d = 2$. (Recall that $c = 3$ here.) Now, $p_0 = 1$ and $p_n = 1/2 + A(1/2)^n + 2(-1)^n$, so $1 = p_0 = 1/2 + A + 2$ and therefore $A = -3/2$. So

$$p_n = \frac{1}{2} - \frac{3}{2}\left(\frac{1}{2}\right)^n + 2(-1)^n.$$

12 (a) The auxiliary equation, $z^2 - 5z + 6 = 0$, has solutions $2, 3$. For a particular solution, substituting $f = ae^x + b$ shows that $a = 1/2$ and $b = 1$. So, for some constants A and B,

$$f(x) = e^x/2 + 1 + Ae^{2x} + Be^{3x}.$$

We then also have $f'(x) = e^x/2 + 2Ae^{2x} + 3Be^{-x}$. The fact that $f(0) = 3/2$ means $A + B + 3/2 = 3/2$, and the fact that $f'(0) = 3/2$ means $1/2 + 2A + 3B = 3/2$, so $A = -1$ and $B = 1$. Therefore, $f(x) = e^{3x} - e^{2x} + e^x/2 + 1$.

(b) There are at least two distinct approaches to finding the inverse: one is to use cofactors, and the other is by row operations. See the Subject Guide for details. The inverse of this matrix turns out to be

$$\begin{pmatrix} -1/2 & 1/2 & -1/2 \\ -2 & 5 & -7 \\ 1 & -2 & 3 \end{pmatrix}.$$

The system of equations is

$$\begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 5 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and since the matrix is invertible, it has the unique solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 5 \\ 2 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & -1/2 \\ -2 & 5 & -7 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a/2 + b/2 - c/2 \\ -2a + 5b - 7c \\ a - 2b + 3c \end{pmatrix}$$

13 (a) The characteristic polynomial factorised quite easily, but some students did not not notice this, and complicated the factorisation of the characteristic polynomial by not spotting the common factor $(2 - \lambda)$. The calculation of eigenvectors was generally well done. A very few students mistakenly offered a zero eigenvector and one or two thought (incorrectly) that $\lambda = 0$ should be omitted from the list of eigenvalues. The eigenvalues were $0, 2, 7$, and examples of corresponding eigenvectors are (respectively)

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}.$$

(b) we separate the variables. On the side where x is the only variable the substitution $u = x^2 + 2x$ leads to the integral $\frac{1}{2}\int du/u$. On the other side (where y is the only variable) the integrand $1/(y^2 - 1)$ needs to be written in partial fraction form, i.e. as $\frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}$. Integrating and writing the constant of integration as $\frac{1}{2}\ln K$, we obtain the solution

$$\frac{|y-1|}{|y+1|} = K(x^2 + 2x).$$

We are told that $y'(1) = 2$, so substituting this value into the left-hand side of the differential equation and of $x = 1$ on the right leads to $y^2 = 4$. Selecting $y = 2$ gives one solution with $K = 1/9$. Selecting $y = -2$ yields another solution with $K = 1$.

(c) the auxiliary equation is $(m - 2)^2 = 0$, so the complementary solution is $y = (At + B)e^{2t}$. In finding a particular solution of the form at^2e^{2t}, it is important to take derivatives carefully by using the product rule. Thus $y' = 2ate^{2t} + 2at^2e^{2t} = (2at + 2at^2)e^{2t}$ and from here $y'' = a(2 + 8t + 4t^2)e^{2t}$. Substitution into the differential equation yields $a = 1$. Thus the general solution is $= (t^2 + At + B)e^{2t}$.

(d) This is an easy question so long as the book-keeping is clean. Let us put $r = 2xy + y^2$, so that $F = xyr^{1/2}$. Now

$$f_x = yr^{1/2} + \frac{1}{2}xyr^{-1/2}(2y)$$

and

$$f_y = xr^{1/2} + \frac{1}{2}xyr^{-1/2}(2x + 2y)$$

Thus we have

$$xf_x + yf_y = r^{1/2}(xy + xy) + \frac{1}{2}xyr^{-1/2}(2xy + 2xy + 2y^2)$$

The first term on the right is $2F$. The second term on the right-hand side simplifies to $xyr^{-1/2}r$ or $xyr^{1/2} = F$. The sum thus amounts to $3F$ as required.

14

The stationarity condition on $P(t)$ yields the equation $P'(t) = e^{-rt}(V'(t) - rV(t)) = 0$. On using $r = 0.04$ and teh expression for $V(t)$, this becomes

$$\frac{1}{2}t^2 - 26t + 26 = 0.$$

Thus the roots are $t = 26 \pm \sqrt{(26^2 - 52)}$. Since $26^2 = 25^2 + 51$, these are $26 \pm \sqrt{25^2 - 1}$. One root, t_+, is therefore just smaller than 51, and the other, t_-, is slightly above 1. The question suggests that we consider the sign of $P'(t)$. Since the exponential factor is always positive we inspect the sign changes of the quadratic factor

$$\frac{1}{2}t^2 - 26t + 26 = \frac{1}{2}(t - t_-)(t - t_+).$$

As t runs from zero upwards, the sign of the quadratic is correspondingly negative, zero, positive, zero, negative thus indicating that the smaller root corresponds to a minimum and the larger to a maximum.

15 In part (a) we follow the hint and find that

$$F_x = 2x(xy+7) + y(x^2+y^2-2) = 0.$$

A similar equation holds with x and y interchanged. Subtracting the two equations, as instructed, gives

$$0 = F_x - F_y = 2(x-y)(xy+7) - (x-y)(x^2+y^2-2)$$

and factorizing yields

$$F_x - F_y = -(x-y)(x^2+y^2-2xy-16) = -(x-y)((x-y)^2-16).$$

Thus either $x = y$, or $x - y = \pm 4$. In the first case we obtain $0 = F_x = 4x^3 + 12x$, so that $x = 0$, or $4x^2 + 12 = 0$; thus we obtain only the solution $x = y = 0$. In the case $x - y = 4$ we obtain, after some calculation, that

$$0 = F_x = 2(2x-4)(x^2-4x+7),$$

so that $x = 2$, since the quadratic factor has no real roots. Thus we obtain $y = -2$. By symmetry, the remaining case $x - y = -4$, i.e. $y - x = 4$, must yield $y = 2$ and $x = -2$. The Hessian matrix $H(x,y)$ is the matrix of second-order partial derivatives:

$$\begin{pmatrix} 6xy+14 & 3x^2+3y^2-2 \\ 3x^2+3y^2-2 & 6xy+14 \end{pmatrix}.$$

Since $det(H(0,0)) = 14^2 - 2^2 > 0$ and $F_{xx} = 14 > 0$ the stationary point is a minimum. Now

$$det(H(2,-2)) = (-24+14)^2 - (24-2)^2 < 0,$$

so this stationary point is a saddle-point. By symmetry, the remaining stationary point also has a negative Hessian determinant, and so is also a saddle-point.

In part (b) the Lagrangian is $L = 2x^{1/2} + 6y^{1/2} - z - \lambda(c + z - x - y)$. The first two first-order conditions show that $x = 1/\lambda^2, y = 9/\lambda^2$ and the third condition gives $\lambda = 1$. Thus $x = 1, y = 9$. But $x + y = c + z$ so $z = 10 - c$ (and we are told that $c < 10$, so this is positive).

16 Part (a) is solved by using an integrating factor; here it is the exponential of

$$\int \frac{2x^2+2}{x}dx = \int (2x + 2/x)dx = x^2 + 2\ln x.$$

Since $e^{2\ln x} = x^2$ the factor is $x^2 e^{x^2}$. We then have

$$yx^2e^{x^2} = \int x^2 e^{x^2} x dx.$$

The easiest way of evaluating the integral is to make the substitution $u = x^2$ and then to use integration by parts. The constant of integration must not be forgotten. A common error was to move from

$$yx^2e^{x^2} = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + c$$

to

$$y = \frac{1}{2} - \frac{1}{2x^2} + c,$$

where c is some constant: this is wrong, since the last term in the expression for y should be $c/(x^2e^{x^2})$, which is not constant.

b This kind of question is easiest computationally when approached by reduction to echelon (or reduced) form provided one aims to keep the arithmetic simple by avoiding the introduction of fractions early in the reduction. There is very little arithmetic involved; all a matter of good book-keeping. Of course, an alternative approach is to use the cofactor method to find the inverse. (Many students did use something like the cofactor method, but misunderstood the definition of cofactors, either not signing them correctly, or mistakenly multiplying them by entries of the matrix.) For the (c) part, since a arises only in the last equation one may solve the first three equations to obtain x, y, z and then deduce from the last equation that $a = 3$ for consistency. A more standard approach is to reduce the following matrix to echelon form.

$$\begin{pmatrix} 1 & 1 & 2 & 5 \\ -1 & -1 & -1 & -4 \\ 3 & -1 & 1 & 6 \\ 1 & 0 & 1 & a \end{pmatrix}.$$

There are many possible ways to perform this reduction. One possible outcome (and there are others possible) is

$$\begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 5-a \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 12-4a \end{pmatrix}.$$

For consistency we require that $12 - 4a = 0$, i.e. $a = 3$. In this case, $z = 1$, $y = 1$ and $x = 2$.

17 Very few students attempted this problem. In part (a) the first step is to obtain f'' by differentiating the first equation. This introduces g', but we may substitute for g' from the second equation. This now gives f'' in terms of f' and g. Now g may be eliminated by using the first equation. The auxiliary equation for f is thus

$$m^2 - m - 2 = 0$$

with roots $m = 2$ and $m = -1$. Thus $f = Ae^{2t} + Be^{-t}$. Since $g = \frac{1}{2}(f' + 2f)$ we have, after some calculation, $g = 2Ae^{2t} + (1/2)Be^{-t}$. These two function will tend to zero precisely when $A = 0$. We are told that $f(0) = 3$ and $g(0) = 3$ so we have two equations $A + B = 3$ and $2A + (1/2)B = 3$, so $A = 1, B = 2$. Thus $f = e^{2t} + 2e^{-t}$ and $g = 2e^{2t} + e^{-t}$. In this case the two functions do not approach zero as t tends to infinity. Without the initial values being given, all that we can say about the initial values of the two functions is that $f(0) = A + B$ and $g(0) = 2A + (1/2)B$. The condition $f(0) = 2g(0)$ is thus equivalent to $A + B = 4A + B$, which means $A = 0$. This is precisely the case when the explosive term Ae^{2t} is absent and both f and g approach zero as t tends to infinity. In part (b) the difference equation

$$y_t - 2y_{t-1} + 2y_{t-2} = 0$$

is solved by first considering the auxiliary equation $m^2 - 2m + 2 = 0$, or $(m-1)^2 + 1 = 0$, so there are no real roots. Setting $u = 1$ and $v = \sqrt{1} = 1$ the relevant two quantities are $r = \sqrt{u^2 + v^2} = \sqrt{2}$ and $\tan^{-1}\frac{v}{u} = \tan^{-1} 1 = \pi/4$. The solution is thus of the form

$$y_t = (\sqrt{2})^t (A\cos(\pi t/4) + B\sin(\pi t/4)).$$

Now $y_0 = 1$, so $A = 1$, and $y_1 = 0$, so $A\cos\pi/4 + B\sin pi/4 = 0$. Hence $A + B = 0$. Thus $B = -1$ and we arrive at

$$y_t = (\sqrt{2})^t (\cos(\pi t/4) - \sin(\pi t/4)).$$

18

(a)
$$A\underline{v} = \begin{pmatrix} 7 & 0 & -3 \\ 1 & 6 & 5 \\ 5 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -14 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$$

$\Rightarrow$ $\underline{v}$ is an eigenvector for eigenvalue $\lambda = 2$

(b)
$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 0 & -3 \\ 1 & 6-\lambda & 5 \\ 5 & 0 & -1-\lambda \end{vmatrix} = (6-\lambda)(\lambda^2 - 6\lambda - 7 + 15) = 0$$
$$= (6-\lambda)(\lambda^2 - 6\lambda + 8)$$
expand by col. 2
$$= (6-\lambda)(\lambda - 2)(\lambda - 4)$$

eigenvalues are $\lambda = 2, 4, 6$

eigenvector for $\lambda = 2$ is $\underline{v}_3 = \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$

for $\lambda = 4$,
$$(A - 4I) = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 5 \\ 5 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{let } \underline{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$
$z = t.$

for $\lambda = 6$,
$$(A - 6I) = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & 5 \\ 5 & 0 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \qquad \text{let } \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 3 \\ -3 & 1 & -7 \\ 1 & 0 & 5 \end{pmatrix} \quad \text{Then } D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad P^{-1}AP = D$$

<u>Check</u>
$$AP = \begin{pmatrix} 7 & 0 & -3 \\ 1 & 6 & 5 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ -3 & 1 & -7 \\ 1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 6 \\ -12 & 6 & -14 \\ 4 & 0 & 10 \end{pmatrix} = PD$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \quad\; \| \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 4\underline{v}_1 \;\; 6\underline{v}_2$$

$$P = \begin{pmatrix} 1 & 0 & 3 \\ -3 & 1 & -7 \\ 1 & 0 & 5 \end{pmatrix}. \qquad |P| = 1(5-3) = 2 \qquad P^{-1} \text{ by cofactors or by:}$$

$$P^{-1}: \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -3 & 1 & -7 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 - r_1]{r_2 + 3r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\underset{\frac{1}{2}r_3}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right) \xrightarrow[r_2 - 2r_3]{r_1 - 3r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & 0 & -3/2 \\ 0 & 1 & 0 & 4 & 1 & -1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

$$\left(\text{check } PP^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ -3 & 1 & -7 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 5/2 & 0 & -3/2 \\ 4 & 1 & -1 \\ -1/2 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right.$$

19 $$P_t - P_{t-1} = \frac{3}{8}\left(3 - P_{t-1} - P_{t-1}\right) - \frac{1}{16}\left(3 - P_{t-2} - P_{t-2}\right)$$

$$P_t - P_{t-1} = \frac{3}{8}\left(3 - 2P_{t-1}\right) - \frac{1}{16}\left(3 - 2P_{t-2}\right)$$

$$\Rightarrow P_t - P_{t-1} = \frac{9}{8} - \frac{6}{8}P_{t-1} - \frac{3}{16} + \frac{1}{8}P_{t-2}$$

$$P_t - \frac{1}{4}P_{t-1} - \frac{1}{8}P_{t-2} = \frac{9}{8} - \frac{3}{16} = \frac{15}{16}$$

auxilary eqn: $\lambda^2 - \frac{1}{4}\lambda - \frac{1}{8} = 0$

$$(\lambda + \tfrac{1}{4})(\lambda - \tfrac{1}{2}) = 0$$

$$\lambda = -\tfrac{1}{4} \text{ or } \lambda = \tfrac{1}{2}$$

constant solution $\quad P^* = \frac{15}{16} \cdot \frac{1}{1 - \frac{1}{4} - \frac{1}{8}} = \frac{15}{16} \cdot \frac{1}{5/8}$

$\Rightarrow P^* = \frac{3}{2} \qquad$ gen'l soln $\left\{ \; P_t = A\left(-\frac{1}{4}\right)^t + B\left(\frac{1}{2}\right)^t + \frac{3}{2}\right.$

if $P_0 = \frac{5}{2} \qquad \frac{5}{2} = A + B + \frac{3}{2} \quad \Rightarrow \quad A + B = 1$

if $P_1 = 1 \qquad 1 = -\frac{1}{4}A + \frac{1}{2}B + \frac{3}{2}$

$$-\tfrac{1}{2} = -\tfrac{1}{4}A + \tfrac{1}{2}B \quad \Rightarrow -A + 2B = -2$$

$$\Rightarrow 3B = -1 \quad \Rightarrow \quad B = -\tfrac{1}{3} \quad \Rightarrow \quad A = \tfrac{4}{3}$$

$$\Rightarrow \quad P_t = \frac{4}{3}\left(-\frac{1}{4}\right)^t - \frac{1}{3}\left(\frac{1}{2}\right)^t + \frac{3}{2}$$

$P_t \to \frac{3}{2}$ as $t \to \infty$, oscillating

20 $g(x) = \ln(1+x) - \sqrt{1+x}$

$g'(x) = \frac{1}{1+x} - \frac{1}{2}(1+x)^{-1/2}$

$g''(x) = \frac{-1}{(1+x)^2} + \frac{1}{4}(1+x)^{-3/2}$

$g(3) = \ln 4 - \sqrt{4} = 2\ln 2 - 2.$

$g'(3) = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^{1/2} = \frac{1}{4} - \frac{1}{4} = 0$

$\Rightarrow x = 3$ is a critical point of g

$g''(3) = -\frac{1}{16} + \frac{1}{4}\frac{1}{(4)^{3/2}} = -\frac{1}{16} + \frac{1}{4}\frac{1}{8} = -\frac{1}{16} + \frac{1}{32}$

$= -\frac{1}{32}$

Taylor expansion is $g(x) \approx \ln 4 - 2 + \left(\frac{-1}{32}\right)\frac{(x-3)^2}{2}$

$= \ln 4 - 2 - \frac{1}{64}(x-3)^2$

21 (a) $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix} = \lambda^2 - \lambda - 20 + 18 = 0$

$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$

eigenvalues are $\lambda = 2$ and $\lambda = -1$.

eigenvectors:

$\lambda = 2$, $(A - 2I) = \begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{v}_1 = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $t \in \mathbb{R}$

$\lambda = -1$, $(A + I) = \begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{v}_2 = t\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $t \in \mathbb{R}$

Set $P = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$, Then $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ $P^{-1}AP = D$

Then $x_t = Ax_{t-1}$, (set $\underline{x}_t = P\underline{z}_t$)

$$X_t = PD^tP^{-1}X_0 = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2^t & 0 \\ 0 & (-1)^t \end{pmatrix}\underbrace{\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\begin{pmatrix} -7 \\ 5 \end{pmatrix}}$$

$\left(P^{-1} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \text{ since } |P| = -1\right)$

$$\Rightarrow \underline{X}_t = -7 \cdot 2^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 5(-1)^t \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_t = 7 \cdot 2^t - 5(-1)^t \\ y_t = -7 \cdot 2^t + 10(-1)^t \end{cases}$$

{ other method possible

$y' = Ay$ set $y = P\underline{z}$ solve $z' = Dz$

P, D as above $\Rightarrow \underline{z} = \begin{pmatrix} \alpha e^{2t} \\ \beta e^{-t} \end{pmatrix}$

$$\underline{y} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \alpha e^{2t} \\ \beta e^{-t} \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

at $t=0$ $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

as above

so $\alpha = -7$, $\beta = 5$

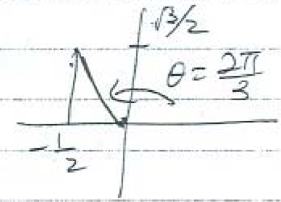
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -7e^{2t} \\ 5e^{-t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x(t) = 7e^{2t} - 5e^{-t} \\ y(t) = -7e^{2t} + 10e^{-t} \end{cases}$$

(b) $Y_t + Y_{t-1} + Y_{t-2} = 18 \qquad y^* = \frac{18}{1+1+1} = 6$

auxiliary equation: $\lambda^2 + \lambda + 1 = 0$

$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$



$r = \sqrt{1} = 1, \quad \cos\theta = \frac{-b}{2r} = -\frac{1}{2}$

$\Rightarrow Y_t = 1^t \left(A \cos \frac{2\pi}{3} t + B \sin \frac{2\pi}{3} t \right) + 6$

$\left. \begin{array}{l} Y_0 = 10 \\ y_1 = 4 \end{array} \right\} \Rightarrow \begin{array}{l} 10 = A + 6 \Rightarrow A = 4 \\ 4 = 4 \cos \frac{2\pi}{3} + B \sin \frac{2\pi}{3} + 6 \\ -2 = 4\left(-\frac{1}{2}\right) + B\left(\frac{\sqrt{3}}{2}\right) \Rightarrow B = 0 \end{array}$

$\Rightarrow Y_t = 6 + 4 \cos \frac{2\pi}{3} t$

, Y_t oscillates between 2 and 10.

$Y_{450} = 6 + 4 \cos\left(\frac{2\pi}{3}(450)\right) = 6 + 4(1) = 10$

22 $Y_t = C_t + I_t = C_t + Y_{t+1}, \qquad 2C_t = Y_t + C_{t-1}$

$\left. \begin{array}{l} C_t = Y_t - Y_{t+1} \\ \quad \text{and} \\ C_{t-1} = Y_{t-1} - Y_t \end{array} \right\}$ substitute these into the above equation

, $2C_t = Y_t + C_{t-1}$

$2(Y_t - Y_{t+1}) = Y_t + Y_{t-1} - Y_t = Y_{t-1}$

write this as $\quad 2Y_{t+1} - 2Y_t + Y_{t-1} = 0$, or

$$Y_t - Y_{t-1} + \frac{1}{2} Y_{t-2} = 0$$

auxiliary equation is $\lambda^2 - \lambda + \frac{1}{2} = 0$
which has complex roots:

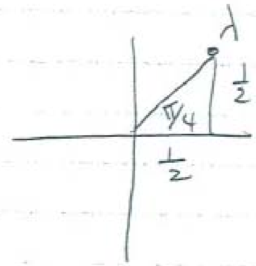
$$\lambda = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1}{2}(1 \pm i)$$

Using $r = \sqrt{c} = \frac{1}{\sqrt{2}}$

$$\cos\theta = \frac{-b}{2r} = \frac{+1}{2(\frac{1}{\sqrt{2}})} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

– or –



$$\Rightarrow Y_t = \left(\frac{1}{\sqrt{2}}\right)^t \left(E\cos\left(\frac{\pi}{4}t\right) + F\sin\left(\frac{\pi}{4}\right)t\right)$$

$Y_0 = 60 \qquad Y_0 = E \quad \Rightarrow \quad E = 60$

$Y_1 = 80 \qquad Y_1 = \frac{1}{\sqrt{2}}\left(60\frac{1}{\sqrt{2}} + F\frac{1}{\sqrt{2}}\right) = 30 + \frac{1}{2}F$

$$\Rightarrow \frac{1}{2}F = 50, \quad F = 100$$

$$Y_t = \left(\frac{1}{\sqrt{2}}\right)^t \left(60\cos\left(\frac{\pi}{4}t\right) + 100\sin\left(\frac{\pi}{4}t\right)\right)$$

as $t \to \infty$, $Y_t \to 0$ oscillating with decreasing magnitude

23 $f(x) = \ln(1+x^3) \qquad x > -1, \quad f(0) = 0$

$f'(x) = \frac{1}{1+x^3} \cdot 3x^2 = 3x^2(1+x^3)^{-1} \qquad f'(0) = 0$

$\Rightarrow x=0$ is a critical point

$f''(x) = 6x(1+x^3)^{-1} - 9x^4(1+x^3)^{-2} \qquad f''(0) = 0$

$f'''(x) = 6(1+x^3)^{-1} - 6x(3x^2)(1+x^3)^{-2} +$ terms in powers of x

$$\Rightarrow f'''(0) = 6$$

$$\Rightarrow \ln(1+x^3) \approx 6x^3/3! = x^3$$

24

$y_t + \frac{5}{3}y_{t-1} - \frac{3}{2}y_{t-2} = 80 \qquad y^* = \frac{80}{1+\frac{5}{3}-\frac{3}{2}} = \frac{80}{\frac{6+10-9}{6}}$

auxiliary equation: $\qquad \Rightarrow y^* = \frac{480}{7}$

$\lambda^2 + \frac{5}{3}\lambda - \frac{3}{2}\lambda = 0 \Leftrightarrow 6\lambda^2 + 10\lambda - 9 = 0$

solutions $\lambda = \frac{-10 \pm \sqrt{100 - 4(6)(-9)}}{2(6)} = \frac{-5 \pm \sqrt{25+54}}{6}$

with ~~roots~~: $\alpha = \frac{-5+\sqrt{29}}{6}$ and $\beta = \frac{-5-\sqrt{29}}{6}$

general solution: $y_t = A\alpha^t + B\beta^t + \frac{480}{7}$

for y_t to approach a finite limit?

$\beta < -1 \quad$ (proof: $\frac{-5-\sqrt{29}}{6} < \frac{-5-8}{6} < -2$)

$0 < \alpha < 1 \quad$ (proof: $\sqrt{29} > 5$ so $-5+\sqrt{29} > 0$; is $\frac{-5+\sqrt{29}}{6} < 1$?

$-5+\sqrt{29} < 6$? $\quad \sqrt{29} < 11$? yes)

so need $B = 0$ in general solution, then $\alpha^t \to 0$ and $y_t \to \frac{480}{7}$.

$y_0 = 100 = A + \frac{480}{7} \Rightarrow A = \frac{220}{7}$

then $y_1 = A\alpha + \frac{480}{7} = \frac{220}{7}\left(\frac{-5+\sqrt{29}}{6}\right) + \frac{480}{7}$

25

$$\frac{dB}{dt} - \frac{1}{20}B = -t-2$$

Linear, with $IF = e^{\int -\frac{1}{20}\,dt} = e^{-t/20}$

So $$Be^{-t/20} = -\int (t+2)e^{-t/20}\,dt$$

$$= 20(t+2)e^{-t/20} - 20\int e^{-t/20}\,dt$$

$$= 20(t+2)e^{-t/20} + 400e^{-t/20} + c$$

$$= 440e^{-t/20} + 20te^{-t/20} + c$$

so $$B = 440 + 20t + ce^{t/20}$$

where $P = B(0) = 440 + c$

so $c = P - 440$ and

$$B = 440 + 20t + (P-440)e^{t/20}$$

$P = 300 \Rightarrow B = 440 + 20t - 140e^{t/20}$

$$\frac{dB}{dt} = 20 - 7e^{t/20}$$

$\frac{dB}{dt} > 0$ when $t = 0$

$\frac{dB}{dt} < 0$ when $t > t^*$ where $7e^{t^*/20} = 20$

$$\left(t^* = 20\ln\left(\tfrac{20}{7}\right)\right)$$

so B reaches a local max at t^* and, ~~since~~ then decreases. (The local max is a global max).

26

$$f = \frac{xe^{2x}}{y^a}$$

$$f_x = \frac{e^{2x} + 2xe^{2x}}{y^a}$$

$$f_y = -\frac{axe^{2x}}{y^{a+1}}$$

$$f_{xx} = \frac{4e^{2x} + 4xe^{2x}}{y^a}$$

$$f_{yy} = +\frac{a(a+1)xe^{2x}}{y^{a+2}}$$

$$3x f_{xx} - xy^2 f_{yy}$$

$$= \frac{12xe^{2x} + 12x^2e^{2x}}{y^a} - \frac{a(a+1)x^2e^{2x}}{y^a}$$

$$= \frac{1}{y^a}\left(12xe^{2x} + \left(12 - a(a+1)\right)x^2e^{2x}\right)$$

This $= 12f$

$$\iff 12 - a(a+1) = 0$$

$$\iff a^2 + a - 12 = 0$$

$$\iff (a+4)(a-3) = 0$$

$$\iff a = -4 \text{ or } 3$$

27 $(D^3 - 2D^2 + D - 2)f = e^{-x}$

$z^3 - 2z^2 + z - 2$

$= (z-2)(z^2+1)$

So $f =$ particular solution $+ Ae^{2x}$

$+ B\cos x + C\sin x$

Try $f = ce^{-x}$.

$(D^3 - 2D^2 + D - 2)f = (-1-2-1-2)ce^{-x}$

$= -6ce^{-x}$

So $c = -\frac{1}{6}$.

$\therefore f = Ae^{2x} + B\cos x + C\sin x - \frac{1}{6}e^{-x}$

28 (a) $A = \begin{pmatrix} 9 & 5 \\ -10 & -6 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 9-\lambda & 5 \\ -10 & -6-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 54 + 50 = 0$

$= \lambda^2 - 3\lambda - 4 = 0$

$= (\lambda - 4)(\lambda + 1) = 0$

The eigenvalues are $\lambda = 4$ and $\lambda = -1$.
For the eigenvectors,

$\lambda = 4$, $A - 4I = \begin{pmatrix} 5 & 5 \\ -10 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \underline{v} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$t \in \mathbb{R}, t \neq 0$

$\lambda = -1$ $\quad A + I = \begin{pmatrix} 10 & 5 \\ -10 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ $\quad v = t\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Let $P = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$, then $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

and $P^{-1}AP = D$. Check $AP = PD$ -

$$AP = \begin{pmatrix} 9 & 5 \\ -10 & -6 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 4 & -2 \end{pmatrix} = PD$$

$$\begin{array}{cc} \underline{v}_1 & \underline{v}_2 \end{array} \qquad \begin{array}{cc} 4\underline{v}_1 & -1\underline{v}_2 \end{array}$$

(b) $\underline{X}_t = A\underline{X}_{t-1}$ has solution $\underline{X}_t = A^t\underline{X}_0$

$\underline{X}_t = PD^tP^{-1}\underline{X}_0$.

$$\underline{X}_t = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4^t & 0 \\ 0 & (-1)^t \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (|P| = -1)$$

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2(4)^t \\ 1(-1)^t \end{pmatrix} = \begin{pmatrix} 2(4)^t - 1(-1)^t \\ -2(4)^t + 2(-1)^t \end{pmatrix}$$

(c) $\underline{w}_t = A\underline{w}_{t-2}$ set $\underline{w}_t = P\underline{u}_t$

Then substituting,

$P\underline{u}_t = A(P\underline{u}_{t-2})$

$u_t = P^{-1}AP\,\underline{u}_{t-2} = Du_{t-2}$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_{t-2} \\ v_{t-2} \end{pmatrix}$$

$\Rightarrow$ $\underline{u}_t = 4\underline{u}_{t-2}$ and $\underline{v}_t = -1\underline{v}_{t-2}$

general solution for u_t:

$\underline{u}_t - 4u_{t-2} = 0$

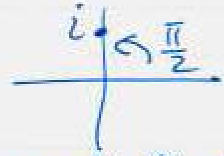
$\lambda^2 - 4 = 0$

$\lambda = \pm 2$

gen'l solution $\left\{ u_t = A(2)^t + B(-2)^t \right.$

general solution for $\underline{V}_t$:

$$\underline{V}_t + \underline{V}_{t-2} = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$



$r = 1,\quad \cos\theta = 0,\quad \theta = \frac{\pi}{2}$

gen'l soln $\left\{ \underline{V}_t = C\cos\frac{\pi}{2}t + D\sin\frac{\pi}{2}t \right.$.

Then $\underline{w}_t = P\underline{u}_t = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} A(2)^t + B(-2)^t \\ C\cos\frac{\pi}{2}t + D\sin\frac{\pi}{2}t \end{pmatrix}$

$$\text{so}\begin{cases} w_t = -A(2)^t - B(-2)^t - C\cos\frac{\pi}{2}t - D\sin\frac{\pi}{2}t \\ z_t = A(2)^t + B(-2)^t + 2C\cos\frac{\pi}{2}t + 2D\sin\frac{\pi}{2}t \end{cases}$$

A, B, C, D constants in $\mathbb{R}$.

29

(a) $f = \cos(nx)\sin(y^2)$

$$f_x = -n\sin(nx)\sin(y^2)$$
$$f_y = 2y\cos(nx)\cos(y^2)$$
$$f_{xx} = -n^2\cos(nx)\sin(y^2)$$
$$f_{yy} = 2\cos(nx)\cos(y^2) - 4y^2\cos(nx)\sin(y^2)$$

$$y^3 f_{xx} - y f_{yy} + f_y = 0$$

$$\Leftrightarrow \quad -n^2y^3\cos(nx)\sin(y^2) - 2y\cos(nx)\cos(y^2) + 4y^3\cos(nx)\sin(y^2) + 2y\cos(nx)\cos(y^2) = 0$$

$$\Leftrightarrow \quad (4-n^2)y^3\cos(nx)\sin(y^2) = 0$$

$$\Leftrightarrow \quad 4-n^2 = 0$$

So $\underline{n = 2}$ (since $n > 0$)

b

$$\frac{7}{q+5} = q-1$$

$$\Rightarrow (q-1)(q+5)=7$$

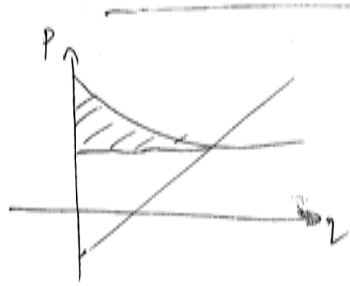
$$\Rightarrow q^2+4q-5-7=0$$

$$\Rightarrow q^2+4q-12=0$$

$$\Rightarrow (q+6)(q-2)=0$$

$$\therefore \; q^*=2$$

$$p^* = q^*-1 = 1$$



$$CS = \int_0^2 \frac{7}{q+5}\,dq - (2\times 1)$$

$$= \Big[7\ln|q+5|\Big]_0^2 - 2 = 7\ln 7 - 7\ln 5 - 2$$

$$= 7\ln\left(\frac{7}{5}\right) - 2$$

30 $(D^3-3D^2+4D-2)f = 4x$.

D^3-3D^2+4D-2 :

z^3-3z^2+4z-2

$z=1 \Rightarrow$ this $=0$ so $(z-1)$ is a factor

$(z-1)(z^2-2z+2)$

roots are $z=1$, $z = 1 \pm \frac{1}{2}\sqrt{-4} = 1 \pm i$.

Complementary function (solution of homogeneous eqn) is

$$Ae^x + e^x(B\sin x + C\cos x)$$

For a particular solution, try $f = ax+b$.

$Df = a$, $D^2f = D^3f = 0$

So: $4a - 2ax - 2b = 4x$

$\therefore$ $4a-2b=0$ $\quad -2a = 4$

$a=-2$, $b=-4$

So, for some A, B, C,

$$f(x) = Ae^x + e^x(B\sin x + C\cos x) - 2x - 4$$

$f(0)=0 \Rightarrow \underline{A+C-4=0}$

$f'(x) = Ae^x + e^x(B\sin x + C\cos x) + e^x(B\cos x - C\sin x) - 2$

$= Ae^x + e^x((B-C)\sin x + (B+C)\cos x) - 2$

$f'(0)=2 \Rightarrow \underline{A+B+C-2=2}$

$f''(x) = Ae^x + e^x((B-C)\sin x + (B+C)\cos x) + e^x((B-C)\cos x - (B+C)\sin x)$

$= Ae^x + e^x(-2C\sin x + 2B\cos x)$

$f''(0)=4 \Rightarrow \underline{A+2B=4}$

So:

$A+C=4$
$A+B+C=4$
$A+2B=4$

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 4 \\ 1 & 0 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So $C=0$ $B=0$ $A=4$

$\underline{f(x) = 4e^x - 2x - 4}$

As $x \to \infty$, $f(x) \to \infty$

31

$$\mathcal{L} = f - \lambda g$$

$$= x + y - \lambda(x^2 - 4xy + y^2 + 2)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 1 - \lambda(2x - 4y) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = 1 - \lambda(-4x + 2y) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 - 4xy + y^2 + 2) = 0 \end{cases}$$

$$\Rightarrow \quad 2x - 4y = -4x + 2y$$

$$\Rightarrow \quad y = x$$

$$x^2 - 4xy + y^2 + 2 = 0 \Rightarrow x^2 - 4x^2 + x^2 + 2 = 0$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

But, given $x \geq 0$, $\underline{x = 1}$, $\underline{y = 1}$

So candidate pair is $(1,1)$.

32 $A_T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & -1 \end{pmatrix}$

$$(A_T|b) = \begin{pmatrix} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & -1 & 1 & -2 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3+R_2]{} \begin{pmatrix} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

let $z=s$, $w=t$

solutions $\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2-s+t \\ 1+s-2t \\ s \\ t \end{pmatrix}$

$$\underline{x} = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \qquad s,t \in \mathbb{R}.$$

33 $|A-\lambda I| = \begin{vmatrix} -6-\lambda & -10 \\ 5 & 9-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 54 + 50 = 0$

$= \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$

The eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -1$.

For $\lambda = 4$,
$(A-4I) = \begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, let $\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda = -1$,
$(A+I) = \begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, let $\underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$, $P^{-1}AP = D$

Check $AP = \begin{pmatrix} -6 & -10 \\ 5 & 9 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 4 & -1 \end{pmatrix} = PD$

$\qquad \qquad \qquad \qquad \overset{\shortparallel}{4\underline{v}_1} \quad \overset{\shortparallel}{-1\underline{v}_2}$

(b) System of difference equations $\underline{x}_t = A\underline{x}_{t-1}$ has solution

$$x_t = PD^tP^{-1}\underline{x}_0 \qquad P = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$|P| = -1+2 = 1 \qquad P^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \qquad P^{-1}\underline{x}_0 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{x}_t = PD^tP^{-1}x_0 = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4^t & 0 \\ 0 & (-1)^t \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4^t \\ -(-1)^t \end{pmatrix} = \begin{pmatrix} -4^t + 2(-1)^t \\ 4^t - (-1)^t \end{pmatrix}$$

(c) (i) Set $y = P\underline{z}$, then $y' = P\underline{z}'$ and $y'' = P\underline{z}''$

substitute in $y'' = Ay$

$$P\underline{z}'' = AP\underline{z} \Rightarrow \underline{z}'' = P^{-1}AP\underline{z} = D\underline{z}$$

$$\begin{pmatrix} z_1'' \\ z_2'' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \begin{cases} z_1'' = 4z_1 \\ z_2'' = -z_2 \end{cases} \qquad \begin{pmatrix} a=4 \\ b=-1 \end{pmatrix}$$

(ii) Solve $z_1'' = 4z_1$, $z_1'' - 4z_1 = 0$

auxiliary equation, $\lambda^2 - 4 = 0$, $\lambda = \pm 2$

general solution is $z_1(x) = Ae^{2x} + Be^{-2x}$, $A, B \in \mathbb{R}$

Solve $z_2'' = -z_2$, $z_2'' + z_2 = 0$

auxiliary equation, $\lambda^2 + 1 = 0$, complex roots, $\lambda = \pm i$ (form $\alpha + i\beta$, $\alpha = 0$, $\beta = 1$)

general solution is $z_2(x) = D\cos x + E\sin x$, $D, E \in \mathbb{R}$

Then
$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} Ae^{2x} + Be^{-2x} \\ D\cos x + E\sin x \end{pmatrix}$$

$$\begin{cases} y_1(x) = -Ae^{2x} - Be^{-2x} - 2D\cos x - 2E\sin x \\ y_2(x) = Ae^{2x} + Be^{-2x} + D\cos x + E\sin x \end{cases}$$

34.

The equilibrium price and quantity are easily seen to be, respectively, 1 and 6. With a per-unit tax of T, we have an equilibrium price p^T given by

$$12(p^T - T) - 6 = 12 - 6p^T.$$

Make sure you understand why. Solving,

$$p^T = 1 + \frac{2}{3}T.$$

Then, the corresponding quantity is

$$q^T = 12 - 6p^T = 6 - 4T.$$

The tax revenue is then Tq^T, because each for each of the q^T units sold, a tax of T is paid. Thus, the tax revenue is $6T - 4T^2$.