05b Revision Problems

1. Expand as a power series, in terms up to x^3, the function
$f(x) = \dfrac{e^{2x}}{(1-x)} = e^{2x}(1-x)^{-1}$.

2. The function $f(x, y, z)$ takes the form

$$f(x,y,z) = \frac{x^{\alpha}y + y^2 z^{\beta}}{\left(xy^{\gamma} + x^4 (yz)^{\delta}\right)^{1/4}}$$

for some numbers $\alpha, \beta, \gamma, \delta$.

If the function is homogeneous of degree 2, determine expressions for β, γ and δ in terms of α.

3. (a) The demand equation for a good is

$$p = 113 - q^2$$

and the supply equation is $p - q^2 - 2q = 1$.

Find the equilibrium price and quantity.

Calculate the consumer surplus and producer surplus at equilibrium.

(b) By separating the variables, find the function $y(t)$ such that $y(0) = 2$ and

$$\frac{dy}{dt} = \frac{-ty^2}{\sqrt{1+t^2}}.$$

4. Suppose the supply and demand functions for a good are, respectively,

$$q^S(p) = 5p - 2, \quad q^D(p) = 12 - 2p.$$

Determine the equilibrium price and quantity. A percentage sales tax of $100r\%$ is imposed. (So, when a consumer buys one unit of the good at a price p, an amount rp is tax.) Find the new equilibrium price and quantity. Find also an expression for the amount of tax revenue.

5 Suppose the demand function for a commodity is given by

$$q = \frac{p}{(p^2+8)^3}.$$

Find the elasticity of demand, in terms of p. Determine the values of p for which the demand is elastic.

6 Expand as a power series, in terms up to x^5, the function

$$f(x) = \ln\left(\frac{1+2x}{1+x}\right).$$

7 The function

$$f(x,y,z) = \frac{x^\alpha y^\beta z^2 + z\sqrt{x^2+y^2}}{(xy+z^\gamma)^\delta}$$

is homogeneous of degree 0. Find the values of γ and δ and derive a relationship between α and β.

[6 marks]

8 The demand equation for a good is $p(q+3) = 20$ and the supply equation is $q - p + 4 = 0$.

Find the equilibrium price and quantity.

Calculate the consumer surplus.

(b) Suppose that the price $p(t)$ of a good varies continuously with time, and that the quantities demanded and supplied are given, respectively, by

$$q^D(p) = 1 - 2p, \quad q^S(p) = 2p.$$

Price adjusts according to the rule

$$\frac{dp}{dt} = \left(q^D(p) - q^S(p)\right)^3.$$

Find an expression for $p(t)$, given that when $t = 0$ the price is $1/8$. How does the price behave as t tends to infinity?

9 (a) Express the following system of equations in matrix form. Then, using a matrix method, show that there is exactly one value of c for which the system has infinitely many solutions. Find all the solutions in this case.

$$\begin{aligned} 2x + y - 3z &= 2 \\ x - y + 2z &= 2 \\ 3x + 3y + cz &= 2. \end{aligned}$$

What are the solutions for other values of c? *[10 marks]*

(b) Find the values of x and y that minimise the function

$$f(x, y) = 8x^2 + 8xy + 12x + 10y^2 + 10y + 20$$

and verify that these values do indeed give a minimum. *[10 marks]*

10 (a) A consumer has utility function $u(x_1, x_2) = x_1 x_2^2$ for two goods, X_1 and X_2. (Here, x_1 and x_2 are, respectively, the amounts of X_1 and X_2 consumed.) Suppose that each unit of X_1 costs $\$p_1$ and each unit of X_2 costs $\$p_2$, and that the consumer has an amount $\$M$ to spend on X_1 and X_2. By using the Lagrange multiplier method, find expressions for the quantities x_1^* and x_2^* that maximise the utility function subject to the budget constraint. *[6 marks]*

What is the corresponding Lagrange multiplier, λ^*? *[2 marks]*

If $V = u(x_1^*, x_2^*)$ is the maximum achievable utility, what is the marginal utility of income, $\dfrac{\partial V}{\partial M}$? *[2 marks]*

(b) The following relationships hold between the sequences C_t, Y_t, I_t:

$$\begin{aligned} C_t &= 10 + \frac{7}{9} Y_{t-1} \\ I_t &= 50 + \frac{2}{9}(Y_{t-1} - Y_{t-2}) \\ Y_t &= C_t + I_t. \end{aligned}$$

Prove that $Y_t - Y_{t-1} + \dfrac{2}{9} Y_{t-2} = 60$. *[5 marks]*

Given that $Y_0 = 271$ and $Y_1 = 270$, find an expression for Y_t. *[5 marks]*

11 (a) A market for a commodity is modelled by taking the demand and supply functions, respectively, as follows:

$$q^D(p) = 1 - p,$$

$$q^S(p) = p,$$

In time period n, for $n \geq 1$, the price p_n is related to the price in the previous period p_{n-1} by the equation:

$$p_n - p_{n-1} = \frac{1}{4}(q^D(p_{n-1}) - q^S(p_{n-1})) + c(-1)^n,$$

where c is a constant.

(i) Show that the price p_n satisfies the equation:

$$p_n - \frac{1}{2}p_{n-1} = \frac{1}{4} + c(-1)^n.$$

[2 marks]

(ii) Show that if $c = 0$, then for some constant A, $p_n = \dfrac{1}{2} + A\left(\dfrac{1}{2}\right)^n$.
[4 marks]

(iii) Suppose that $p_0 = 1$. Solve the equation when $c = 3$ by substituting for p_n the expression for p_n obtained in (ii), plus a term of the form $d(-1)^n$.

[4 marks]

12 (a) Find the function $f(x)$ satisfying

$$f(0) = 3/2, \quad \frac{df}{dx}(0) = 3/2, \quad \frac{d^2 f}{dx^2} - 5\frac{df}{dx} + 6f = e^x + 6.$$

(b) Find the inverse of the matrix

$$\begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 5 \\ 2 & 1 & 3 \end{pmatrix}$$

Use your result to find the solution to the system of equations

$$\begin{aligned} -2x + y + 2z &= a \\ 2x + 2y + 5z &= b \\ 2x + y + 3z &= c, \end{aligned}$$

where a, b, c are any numbers.

13 **a** Find the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 3 \\ 0 & 3 & 2 \end{pmatrix}$$

and obtain one eigenvector for each eigenvalue.

b (a) Find a function $y(x)$ such that $y'(1) = 2$ and

$$\frac{dy}{dx} = \frac{(x+1)(y^2-1)}{2x+x^2}.$$

c Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2e^{2t}.$$

For a particular solution try $y = at^2e^{2t}$.

d Suppose that $F(x,y) = xy\sqrt{(2xy + y^2)}$. Show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 3F(x,y).$$

14 An asset has just been purchased for one million dollars. It is believed that the asset may be sold in t years time for a value $V(t)$ (measured in million dollars) given by

$$V(t) = \frac{1}{2}\left(1 + (t-1)^2\right).$$

Assuming an average annual interest rate of $r = 4\%$ this value has a present value of

$$P(t) = e^{-0.04t}V(t),$$

measured in millions of dollars of money of the present time. Show that $P(t)$ has two stationary points, one of which is very nearly $t = 1$ and the other very nearly $t = 51$. By considering the sign of $P'(t)$, show that the best time to sell the asset is very nearly $t = 51$ years.

15 (a) Find, using the product rule, the first-order partial derivatives, of the function

$$f(x, y) = (x^2 + y^2 - 2)(xy + 7).$$

Hence determine the stationary points of the function.(You will find it helpful to consider the difference of the two first-order partial derivatives.) Classify all the stationary points.

(b) Use the Lagrange Multiplier Method to maximize

$$2\sqrt{x} + 6\sqrt{y} - z$$

subject to

$$x + y = c + z,$$

for $x, y, z \geq 0$, where c is a *positive* constant and $c < 10$.

16 (a) Use an integrating factor to solve the differential equation

$$\frac{dy}{dx} + 2\frac{x^2+1}{x}y = x.$$

(b) Find the inverse of the matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

c Determine the value of the constant a for which the following system of equations has a solution, and find the solution when a takes this value.

$$\begin{aligned} x+y+2z &= 5 \\ -x-y-z &= -4 \\ 3x-y+z &= 6 \\ x+z &= a. \end{aligned}$$

17 (a) The functions $f(t)$ and $g(t)$ are related as follows:

$$\begin{aligned} \frac{df}{dt} &= -2f(t)+2g(t) \\ \frac{dg}{dt} &= -2f(t)+3g(t). \end{aligned}$$

Show that

$$\frac{d^2f}{dt^2}-\frac{df}{dt}-2f=0.$$

Hence find $f(t)$ and $g(t)$ if $f(0)=3$ and $g(0)=3$.

Show that $f(t)$ and $g(t)$ will approach 0 as t tends to infinity precisely when the initial values of f and g satisfy $f(0)=2g(0)$.

(b) Find the sequence y_t such that $y_0=1, y_1=0$ and

$$y_t-2y_{t-1}+2y_{t-2}=0.$$

18

Let

$$\mathbf{A}=\begin{pmatrix} 7 & 0 & -3 \\ 1 & 6 & 5 \\ 5 & 0 & -1 \end{pmatrix} \qquad \mathbf{v}=\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$$

(a) Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

(b) Find the other eigenvalues of $\mathbf{A}$. Hence find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$.

Check that $\mathbf{AP}=\mathbf{PD}$.

Find $\mathbf{P}^{-1}$.

19 A market for a commodity is modelled by the supply and demand functions defined as follows:

$$q^S(p) = p\,, \qquad q^D(p) = 3 - p$$

The price change $p_t - p_{t-1}$ depends on the excess demand in the previous two periods according to the equation:

$$p_t - p_{t-1} = \frac{3}{8}\Big(q^D(p_{t-1}) - q^S(p_{t-1})\Big) - \frac{1}{16}\Big(q^D(p_{t-2}) - q^S(p_{t-2})\Big).$$

Find a formula for p_t, given that $p_0 = \frac{5}{2}$ and $p_1 = 1$.

Describe in words the behaviour of p_t as t increases.

(**20** Find the Taylor expansion about $x = 3$ of the function

$$g(x) = \ln(1+x) - \sqrt{1+x}$$

up to and including the quadratic term.

21 (a) (i) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

(ii) Find explicit formulae for the sequences x_t, y_t that satisfy the system of recurrence equations

$$\begin{aligned} x_t &= 5x_{t-1} + 3y_{t-1} \\ y_t &= -6x_{t-1} - 4y_{t-1} \end{aligned}$$

and the initial conditions $x_0 = 2$, $y_0 = 3$.

(iii) Find the functions $x(t)$, $y(t)$ that satisfy the system of differential equations

$$\begin{aligned} x_1'(t) &= 5x(t) + 3y(t) \\ y_1'(t) &= -6x(t) - 4y(t) \end{aligned}$$

and the initial conditions $x(0) = 2$, $y(0) = 3$.

(b) If $Y_0 = 10$ and $Y_1 = 4$, find an expression for the solution Y_t of the equation

$$Y_t + Y_{t-1} + Y_{t-2} = 18.$$

Describe carefully the behaviour of Y_t as $t \to \infty$.

Find the value of Y_{450}.

22 (a) Suppose that consumption this year, C_t, is the average of this year's income, Y_t, and last year's consumption and that next year's income equals current investment, I_t; that is,

$$C_t = \frac{1}{2}(Y_t + C_{t-1}) \qquad \text{and} \qquad Y_{t+1} = I_t\ .$$

Assuming the usual equilibrium condition, $Y_t = C_t + I_t$, show that Y_t satisfies the following equation,

$$Y_t - Y_{t-1} + \frac{1}{2}Y_{t-2} = 0\ .$$

Given the initial conditions $Y_0 = 60$ and $Y_1 = 80$, find an expression for Y_t which satisfies this equation.

Describe the long term behaviour of Y_t as $t \to \infty$.

23 Find the Taylor series expansion about $x = 0$ of the function

$$f(x) = \ln(1 + x^3)$$

up to and including terms of degree 3 in x.

24 Find the general solution of

$$y_{t+2} + \frac{5}{3}y_{t+1} - \frac{3}{2}y_t = 80 \qquad t = 0, 1, 2, \ldots .$$

If $y_0 = 100$, determine a value of y_1 so that y_t approaches a finite limit as $t \to \infty$, and state what this finite limit is.

25

(a) The balance $B(t)$ of a bank account at time t is subject to continuous compounding and a net outflow and satisfies the differential equation

$$\frac{dB}{dt} = \frac{B(t)}{20} - t - 2.$$

The balance at time 0 is P. By solving this differential equation, find $B(t)$.

Suppose that $P = 300$. Show that the balance $B(t)$ initially increases, to a maximum value, and thereafter decreases.

26 The function $f(x, y)$ is defined for $x, y > 0$ by

$$f(x, y) = \frac{xe^{2x}}{y^a},$$

ıere a is a fixed real number.

nd expressions for the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}.$$

termine the values of a for which the function will satisfy the equation

$$3x\frac{\partial^2 f}{\partial x^2} - xy^2\frac{\partial^2 f}{\partial y^2} = 12f.$$

27 Find the general solution to the differential equation

$$\frac{d^3 f}{dx^3} - 2\frac{d^2 f}{dx^2} + \frac{df}{dx} - 2f = e^{-x}.$$

28 (a) Find the eigenvalues and the corresponding eigenvectors for the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 9 & 5 \\ -10 & -6 \end{pmatrix}.$$

Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. Check that $\mathbf{AP} = \mathbf{PD}$.

(b) Consider the system of recurrence equations $\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1}$,

$$\begin{aligned} x_t &= 9x_{t-1} + 5y_{t-1} \\ y_t &= -10x_{t-1} - 6y_{t-1} \end{aligned} \qquad \mathbf{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

Find expressions for the sequences x_t and y_t which satisfy these equations and the initial conditions $x_0 = 1$, $y_0 = 0$.

(c) Now consider the system of **second order** recurrence equations $\mathbf{w}_t = \mathbf{A}\mathbf{w}_{t-2}$,

$$\begin{aligned} w_t &= 9w_{t-2} + 5z_{t-2} \\ z_t &= -10w_{t-2} - 6z_{t-2} \end{aligned} \qquad \mathbf{w}_t = \begin{pmatrix} w_t \\ z_t \end{pmatrix}.$$

Set $\mathbf{w}_t = \mathbf{P}\mathbf{u}_t$ where $\mathbf{P}$ is the matrix you found in part (a), to define new sequences u_t and v_t such that

$$\mathbf{w}_t = \begin{pmatrix} w_t \\ z_t \end{pmatrix} = \mathbf{P}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbf{P}\mathbf{u}_t.$$

Show that the sequences u_t and v_t satisfy second-order recurrence equations of the form

$$u_t = au_{t-2}, \qquad v_t = bv_{t-2}$$

for some constants a and b. Find the numbers a and b and write down the equations. Find a general solution of each of these equations (sequences u_t and v_t).

Hence, or otherwise, find a general solution of $\mathbf{w}_t = \mathbf{A}\mathbf{w}_{t-2}$ (sequences w_t and z_t).

29

(a) The function $f(x, y)$ is given by

$$f(x, y) = \cos(nx)\sin(y^2),$$

where n is a positive integer. Find

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}.$$

Find the value of n if f satisfies

$$y^3 \frac{\partial^2 f}{\partial x^2} - y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = 0.$$

(b) The inverse demand function for a product is

$$p = \frac{7}{q+5}$$

and the inverse supply function is

$$p = q - 1.$$

Here, p denotes price and q denotes quantity. Find the equilibrium price and quantity, and calculate the consumer surplus.

30 Consider the following differential equation:

$$\frac{d^3f}{dx^3} - 3\frac{d^2f}{dx^2} + 4\frac{df}{dx} - 2f = 4x.$$

(i) Find the general solution of the differential equation.

(ii) Find the function f which satisfies the differential equation and which is such that

$$f(0) = 0, \quad f'(0) = 2 \text{ and } f''(0) = 4.$$

Describe the behaviour of this particular solution as x tends to infinity.

31

Use the method of Lagrange multipliers to find possible maximum or minimum points of the function

$$f(x, y) = x + y$$

subject to the constraints

$$x^2 - 4xy + y^2 = -2, \qquad x \geq 0, \quad y \geq 0.$$

32 Consider the following system of equations, where λ is a constant, $\lambda \in \mathbb{R}$,

$$\begin{cases} x + 2y + z = 0 \\ 3x + 5y + 2z = 1 \\ y + \lambda z = 2. \end{cases}$$

(i) Find all values of λ for which the system of equations is consistent.
If the system is consistent, is the solution unique? Justify your answer.

(ii) In the case(s) that the above system has a unique solution, solve the system using any matrix method (Gaussian elimination, Cramer's rule, or inverse matrix) and hence write down expressions for x, y, z in terms of λ.

33 Find the eigenvalues and the corresponding eigenvectors for the matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{pmatrix} -6 & -10 \\ 5 & 9 \end{pmatrix}$$

Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. Check that $\mathbf{AP} = \mathbf{PD}$.

(b) Consider the system of difference equations $\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1}$,

$$\begin{aligned} x_t &= -6x_{t-1} - 10y_{t-1} \\ y_t &= 5x_{t-1} + 9y_{t-1} \end{aligned} \qquad \mathbf{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

Find expressions for the sequences x_t and y_t which satisfy these equations and the initial conditions $x_0 = 1,\ y_0 = 0$.

(c) Suppose that the functions $y_1(x)$ and $y_2(x)$ satisfy the **second order** differential equations $\mathbf{y}'' = \mathbf{A}\mathbf{y}$,

$$\begin{aligned} \frac{d^2y_1}{dx^2} &= -6y_1 - 10y_2 \\ \frac{d^2y_2}{dx^2} &= 5y_1 + 9y_2\,. \end{aligned}$$

Set $\mathbf{y} = \mathbf{Pz}$ where $\mathbf{P}$ is the matrix you found in part (a), to define new functions $z_1(x)$ and $z_2(x)$,

$$\mathbf{y} = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \mathbf{P}\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} = \mathbf{Pz}.$$

(i) Show that the functions $z_1(x)$ and $z_2(x)$ satisfy second-order differential equations of the form

$$\frac{d^2z_1}{dx^2} = az_1\,, \qquad \frac{d^2z_2}{dx^2} = bz_2$$

for some constants a and b. Find the numbers a and b and write down the equations.

(ii) Solve these equations for $z_1(x)$ and $z_2(x)$. Hence, or otherwise, find a general solution for $y_1(x)$ and $y_2(x)$.

34 Suppose the supply and demand functions for a good are, respectively,

$$q^S(p) = 12p - 6, \quad q^D(p) = 12 - 6p.$$

Determine the equilibrium price and quantity. A per-unit (or excise) tax of T is imposed. Find the new equilibrium price and quantity. Find also an expression for the amount of tax revenue.