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Numerical Integration
Tutoring Sheet #6 – Solution

1. Obtain the expansion of the following functions as indicated :

a. f(x) = $e^{\frac{x}{2}}$ according to powers of x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \ldots$$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \ldots$$

b. f(x) = lnx according to powers of x - 2 ;using Taylor's:

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots$$

f(2) = ln2

f'(x) = x^{-1} $\Rightarrow$ f'(2) = ½

f''(x) = $-x^{-2}$ $\Rightarrow$ f''(2) = - ¼

f'''(x) = $2x^{-3}$ $\Rightarrow$ f'''(2) = ¼

$$\ln x = \ln 2 + \frac{x-2}{2} - \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{1}{4}\frac{(x-2)^3}{3!} - \ldots$$

c. f(x) = $\cos^2 x$ according to powers of x

$$\cos^2 x = 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \ldots$$

d. f(x) = $\frac{1}{1+x^2}$ according to powers of x

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x (1 - x^2 + x^4 - x^6 + ...)dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$$

2. Use the expansions of e^{ix}, cosx and sinx to show that :

e^{ix} = cosx + isinx

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} .....$$

$$\Rightarrow e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + ...$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + ...$$

$$= (1 - \frac{x^2}{2!} + + \frac{x^4}{4!} - \frac{x^6}{6!} + ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)$$

= cosx + isinx

3. Evaluate using expansion, the following integral : $\int_0^1 \frac{\sin x}{x}$ dx

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \text{ ....} ; \quad \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \text{ ....}$$

$$\int_0^1 \frac{\sin x}{x} dx = \int_0^1 (1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^5}{7!} + ...)dx = x - \frac{x^3}{3\times 3!} + \frac{x^5}{5\times 5!} - \frac{x^7}{7\times 7!} ... \Big|_0^1$$

= 0.946

4. Using the expansions of e^x and sinx , cosx ,find the expansions of the following:

a. $e^{1-\sin x} = e^{-1}e^{-\sin x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...; \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ......$$

$-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}$ ...... replace this by x in e^x :

$$e^{-\sin x} = 1 + (-x + \frac{x^3}{3!} ...) + \frac{1}{2!}(-x + \frac{x^3}{3!} ...)^2 + \frac{1}{3!}(-x + \frac{x^3}{3!} ...)^3$$

$$= 1 - x + \frac{x^3}{3!} \;.....+ \frac{1}{2!}(x^2 - 2\frac{x^4}{3!}...) + \frac{1}{3!}[(-x)^3....]$$

$$= 1 - x + \frac{1}{2!}x^2 - \frac{x^4}{3!}.......$$

$$e^{1-\sin x} = e^{-1}e^{-\sin x} = \frac{1}{e}\left(1 - x + \frac{1}{2!}x^2 - \frac{x^4}{3!}.......\right)$$

b. $e^x \cos x$

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}.....\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.....$$

$$+x - \frac{x^3}{2!} + \frac{x^5}{4!}.....$$

$$+ \frac{x^2}{2!} - \frac{x^4}{2!2!} + \frac{x^6}{2!4!}....$$

$$+ \frac{x^3}{3!} - \frac{x^5}{3!2!} + \frac{x^7}{3!4!}$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30}.........$$

5. Use simpson's rule as indicated to evaluate the following integrals:

a. $\int_0^1 e^{-x^2}dx$ with 5 ordinates = 0.747

b.) $\int_1^3 \ln x dx$ with 9 ordinates = 1.2958 ;

Compare your answer with a precise answer obtained by integration by parts or otherwise:

$$\int_1^3 \ln x dx = x\ln x - x \Big|_1^3 = 3\ln 3 - 3 - (1\ln 1 - 1) = 3\ln 3 - 2 = 1.2958$$