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Differential Equations
Tutoring Sheet #16 – Solution

1. Solve the following Differential Equations:

a. $y'' = xe^x$ integrating : $y' = xe^x - e^x + c_1$ (integration by parts)
integrating once more : $y = xe^x - 2e^x + c_1x + c_2$

b. $2\sqrt{x}\dfrac{dy}{dx} = x^2 - 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^2-1}{2\sqrt{x}} \Rightarrow y = \int \dfrac{x^2-1}{2\sqrt{x}}dx = \dfrac{1}{2}\int (x^2-1)x^{\frac{-1}{2}}dx$

$y = \dfrac{1}{2}\int (x^{\frac{3}{2}} - x^{\frac{-1}{2}})dx = \dfrac{1}{2}(\dfrac{2x^{\frac{5}{2}}}{5} - 2\sqrt{x}) + c = \dfrac{x^{\frac{5}{2}}}{5} - \sqrt{x} + c$

c. $(2x + 3y)dx + (y - x)dy = 0$ homogeneous of degree 1:

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{2x+3y}{y-x} \Rightarrow \dfrac{dy}{dx} = -\dfrac{2+3\frac{y}{x}}{\frac{y}{x}-1} = -\dfrac{2+3v}{v-1}$ Let $y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$;substitute in the eq.: $v + x\dfrac{dv}{dx} = -\dfrac{2+3v}{v-1}$

Separable: $-\dfrac{dx}{x} = \dfrac{v-1}{v^2+2v+2}dv$ integrating: $-\ln|x| = \int \dfrac{v-1}{v^2+2v+2}dv$

$= \int \dfrac{v-1}{(v+1)^2+1}dv = \int \dfrac{v}{(v+1)^2+1}dv - \int \dfrac{1}{(v+1)^2+1}dv$

Let $t = v + 1$; $dt = dv$ and $v = t-1$

$= \int \dfrac{t-1}{t^2+1}dt - \int \dfrac{1}{t^2+1}dt = \int \dfrac{t}{t^2+1}dt - 2\int \dfrac{1}{t^2+1}dt = \frac{1}{2}\ln(t^2+1) - 2\tan^{-1}t + c$

$-\ln|x| = \frac{1}{2}\ln(v^2+2v+2) - 2\tan^{-1}(v+1) + c$

$-\ln|x| = \frac{1}{2}\ln(y^2/x^2 + 2y/x + 2) - 2\tan^{-1}(y/x + 1) + c$

d. $y\dfrac{dy}{dx}=\sqrt{y^2+1} \Rightarrow \dfrac{ydy}{\sqrt{y^2+1}}=dx$ integrating (let $u=y^2+1$; $ydy=$ ½ du)

$x = -\sqrt{y^2+1} + c$

e. $x^3\,dx + (y+1)^2\,dy = 0$;Separable : $(y+1)^2\,dy = -x^3\,dx$

$4(y+1)^3 + 3x^4 + C = 0$

f. $\dfrac{dy}{dx}=\dfrac{y(x+2y)}{x(2x+y)}$ Homogeneous of degree 2 ;dividing by x^2 and

letting y = vx : $v+x\dfrac{dv}{dx}=\dfrac{v+2v^2}{2+v} \Rightarrow x\dfrac{dv}{dx}=\dfrac{v^2-v}{2+v} \Rightarrow \dfrac{2+v}{v^2-v}dv=\dfrac{dx}{x}$

$\Rightarrow \int\dfrac{2+v}{v^2-v}dv=\int\dfrac{dx}{x} \Rightarrow \int\dfrac{2+v}{v(v-1)}dv=\int\dfrac{dx}{x}$ using partial fractions:

$\dfrac{2+v}{v(v-1)} = \dfrac{a}{v}+\dfrac{b}{v-1} \Rightarrow 2+v = a(v-1) + bv$

Choose $v = 1 \Rightarrow 3 = a(0) + b \Rightarrow b = 3$

Choose $v = 0 \Rightarrow 2 = a(-1) + b(0) \Rightarrow a = -2$

$\Rightarrow \int\dfrac{-2}{v}dv+\int\dfrac{3}{v-1}dv = \int\dfrac{dx}{x} \Rightarrow -2\ln|v| + 3\ln|v-1| = \ln|x| + C$

$\Rightarrow -2\ln|y/x| + 3\ln|y/x - 1| = \ln|x| + C$

2. Solve the following differential equations:

a. $\dfrac{dy}{dx}+2y=8x^2-2$ First order linear : P = 2 ; Q = $8x^2 - 2$

y= $e^{\int -Pdx}\left(\int Qe^{\int Pdx}\,dx+c\right)$ **=** $e^{\int -2dx}\left(\int(8x^2-2)e^{\int 2dx}\,dx+c\right)$

y= $e^{-2x}\left(\int(8x^2-2)e^{2x}\,dx+c\right)$; $\int(8x^2-2)e^{2x}\,dx$ By Parts :

$u = 8x^2 - 2 \Rightarrow du = 16x\,dx$; $dv = e^{2x}\,dx \Rightarrow v = ½\, e^{2x}$

$\int udv=uv-\int vdu$; $\int(8x^2-2)e^{2x}\,dx = ½\,(8x^2 - 2)\,e^{2x} - ½\int 16xe^{2x}dx$

By Parts again: $u = x \Rightarrow du = dx$; $v= e^{2x}\,dx \Rightarrow v = ½\, e^{2x}$

$\int xe^{2x}dx = xe^{2x} - \int e^{2x}dx = xe^{2x} - ½\, e^{2x}$

$\int(8x^2-2)e^{2x}\,dx = ½\,(8x^2 - 2)\,e^{2x} - 8\,xe^{2x} + 4e^{2x}$

$y = e^{-2x}\left(½\,(8x^2-2)\,e^{2x} - 8\,xe^{2x} + 4e^{2x} + c\right) = ½\,(8x^2-2) - 8x + 4 + ce^{-2x}$

$y = 4x^2 - 8x + 3 + ce^{-2x}$

b. $x\dfrac{dy}{dx}+3y=2x+5 \Rightarrow \dfrac{dy}{dx}+\dfrac{3}{x}y=\dfrac{2x+5}{x}$;First order linear:P=$\dfrac{3}{x}$;Q=$\dfrac{2x+5}{x}$

y= $e^{\int -\frac{3}{x}dx}\left(\int \frac{2x+5}{x}e^{\int \frac{3}{x}dx}\,\mathrm{dx}+c\right)$ = $e^{3\ln x}\left(\int \frac{2x+5}{x}e^{3\ln x}\,\mathrm{dx}+c\right)$

y= $x^{3}\left(\int \frac{2x+5}{x}\times x^{3}\,\mathrm{dx}+c\right)$; **Recall :** $e^{-3\ln x}=e^{\ln x^{-3}}=x^{-3}$

y = $x^{-3}\left(\int (2x+5)x^{2}\,\mathrm{dx}+c\right)$ = $x^{-3}\left(\int (2x^{3}+5x^{2})\,\mathrm{dx}+c\right)$

y = x^{-3} (½ x^4 + 5/3 x^3 + c) = ½ x + 5/3 + cx^{-3}

c. $x^2\frac{dy}{dx}+xy+y=0 \Rightarrow x^2\frac{dy}{dx}+(x+1)y=0 \Rightarrow \frac{dy}{dx}+(\frac{x+1}{x^2})y=0$

First order linear:P= $\frac{x+1}{x^2}$;Q= 0 ; **y=** $e^{\int \frac{x+1}{x^2}dx}\left(\int (0)e^{\int \frac{x+1}{x^2}dx}\,\mathrm{dx}+c\right)$

y= $ce^{\int\left(-\frac{1}{x}-\frac{1}{x^2}\right)dx}$ = c $e^{-\ln x+\frac{1}{x}}$ = c($e^{\ln x^{-1}}\times e^{\frac{1}{x}}$) = (c/x) $e^{\frac{1}{x}}$

Another method: Separable $\frac{dy}{y}=-\frac{x+1}{x^2}dx$ then integrate.

d. $\frac{dy}{dx}=(x+y)^2$

Let u = x + y ⇒ y = u – x ⇒ $\frac{dy}{dx}=\frac{du}{dx}-1$

$\frac{dy}{dx}=(x+y)^2 \Rightarrow \frac{du}{dx}-1=u^2 \Rightarrow \frac{du}{1+u^2}=dx \Rightarrow$ $\tan^{-1}$ u = x + c

$\tan^{-1}$ (x+y) = x + c

e. $\frac{d^2y}{dx^2}$ - $\frac{dy}{dx}$ - 6y =3x^2 + x + 2 If y = 1 and $\frac{dy}{dx}$ = 1 when x= 0

Auxiliary Equation: $r^2-r-6=0 \Rightarrow$ r = - 2 ; r = 3

y_c = $Ae^{-2x}+Be^{3x}$

y_p = C +Dx+Ex^2 to substitute this in the equation, we need y_p' and y_p''

y_p' =D+2Ex ;y_p'' = 2E , Substituting in the equation:

2E – D – 2Ex - 6(C +Dx+Ex^2) = 3x^2 + x + 2

-6Ex^2 +(-2E -6D)x + 2E – D - 6C = 3x^2 + x + 2

-6E = 3 ⇒ E=- ½; -2E -6D=1⇒ D = 0;2E –D -6C=2 ;C=-½

y_p = - ½ x^2 – ½ ; General solution : $\mathbf{y = y_c + y_p}$

y = $Ae^{-2x}+Be^{3x}$ - ½ x^2 – ½

y(0) = 1 ⇒ $Ae^{0}+Be^{0}$ - ½ (0) – ½ = 1 ⇒ A + B = 3/2 y' = $-2Ae^{-2x} + 3Be^{3x}$ - x

y'(0) = 1 $\Rightarrow$ $-2Ae^{0} + 3Be^{0}$ - 0 = 1 $\Rightarrow$ -2A + 3B = 1

Solving simultaneously for A & B : A = 7/10 ; B = 4/5

y = $\frac{7}{10}e^{-2x} + \frac{4}{5}e^{3x}$ - ½ x^2 – ½

f. $\frac{d^2y}{dx^2}$ - $\frac{dy}{dx}$ - 2y = e^{2x}

Auxiliary Equation: $r^2 - r - 2 = 0 \Rightarrow$ r = - 1 ; r = 2

y_c = $Ae^{-x} + Be^{2x}$

Note: e^{2x} is part of y_c

The particular solution ~~y_p = Ce^{2x}~~ will not work!

y_p = Ce^{2x} to substitute this in the equation, we need y_p' and y_p''

y_p' = $2Ce^{2x}$; y_p'' = $4Ce^{2x}$, Substituting in the equation:

$4Ce^{2x} - 2Ce^{2x} - 2Ce^{2x} = e^{2x} \Rightarrow 0e^{2x} = e^{2x}$??

To fix it we attach x to Ce^{2x}:

Let y_p = Cxe^{2x}; y_p' = $(2Cx+C)e^{2x}$; y_p'' = $(4Cx+4C)e^{2x}$

$(4Cx+4C)e^{2x}$ - $(2Cx+C)e^{2x}$ - $2Cxe^{2x}$ = e^{2x}

$3Ce^{2x} = e^{2x} \Rightarrow$ C = 1/3 $\Rightarrow$ y_p = 1/3 xe^{2x} ;

y = $Ae^{-x} + Be^{2x}$ + $(1/3)xe^{2x}$

Example on the normal case: $\frac{d^2y}{dx^2}$ - 6$\frac{dy}{dx}$ + 9y = e^{2x}

Auxiliary Equation: $r^2 - 6r + 9 = 0 \Rightarrow$ r = 3 ; r = 3

y_c = $(A + Bx)e^{3x}$

y_p = Ce^{2x} to substitute this in the equation, we need y_p' and y_p''

y_p' = $2Ce^{2x}$; y_p'' = $4Ce^{2x}$, Substituting in the equation:

$4Ce^{2x} - 12Ce^{2x} + 9Ce^{2x} = e^{2x} \Rightarrow Ce^{2x} = e^{2x} \Rightarrow$ C = 1 $\Rightarrow$ y_p = e^{2x}

y = $(A + Bx)e^{3x}$ + e^{2x}

g. $\frac{d^2y}{dx^2}$ - 2$\frac{dy}{dx}$ - y = sin2x

Auxiliary Equation: $r^2 - 2r - 1 = 0 \Rightarrow$ r = $1 + \sqrt{2}$; r = $1 - \sqrt{2}$

y_c = $Ae^{(1-\sqrt{2})x} + Be^{(1+\sqrt{2})x}$

y_p = C cos2x + Dsin2x ; y_p' = -2Csin2x + 2Dcos2x

y_p'' = -4Ccos2x -4Dsin2x , Substituting in the equation:

-4Ccos2x -4Dsin2x +4Csin2x - 4Dcos2x -C cos2x - Dsin2x = sin2x

(-4D + 4C – D)sin2x + (-4C -4D – C)cos2x = sin2x

4C – 5D = 1 ; 5C + 4D = 0 Solving simultaneously for C & D:
C = 4/41 ; D = -5/41 ; y_p = 4/41 cos2x – 5/41 sin2x
y = $Ae^{(1-\sqrt{2})x} + Be^{(1+\sqrt{2})x}$ + 4/41 cos2x – 5/41 sin2x

h. $\frac{d^2y}{dx^2}$ + y = sinx + cosx

Auxiliary Equation: $r^2 + 1 = 0 \Rightarrow$ r = $\pm i$

y_c = $e^{\frac{-a}{2}x}(A\cos\alpha\, x + B\sin\alpha\, x)$; $\alpha = \frac{\sqrt{4b-a^2}}{2}$ = $\frac{\sqrt{4(1)-0^2}}{2}$ = 1

y_c = $e^{\frac{-0}{2}x}(A\cos x + B\sin x)$; y_c = $A\cos x + B\sin x$

~~y_p = Csinx + Dcosx~~ **will not work since it's part of y_c**

Let y_p = x(Csinx + Dcosx) ; y_p' = Csinx + Dcosx +x(C cosx-Dsinx)

y_p'' = C cosx – Dsinx + C cosx – Dsinx + x(-Csinx – Dcosx)

Substituting in the equation:
2C cos x – 2Dsinx – Cxsinx –Dxcosx + Cxsinx + Dxcosx = sinx + cosx
2C = 1 ⇒ C = ½ ; -2D = 1 ⇒ D = - ½ ⇒ y_p = ½ x(sinx – cosx)
y = $A\cos x + B\sin x$ + ½ x(sinx – cosx)

i. $\frac{d^2y}{dx^2}$ + $\frac{dy}{dx}$ - 6y = e^{2x} cosx

Auxiliary Equation: $r^2 + r - 6 = 0 \Rightarrow$ r = 2 ; r = -3
y_c = $Ae^{-3x} + Be^{2x}$
y_p = e^{2x}(Csinx + Dcosx)
y_p' = $2e^{2x}$(Csinx+Dcosx)+e^{2x}(Ccosx–Dsinx)
= e^{2x}[(2C-D)sinx+(C+2D)cosx];
y_p'' = $2e^{2x}$[(2C-D)sinx+(C+2D)cosx]+e^{2x}[(2C-D)cosx–(C+2D)sinx]
= e^{2x} [(3C-4D)sinx + (4C+3D)cosx]
substituting in the equation:
e^{2x} [(3C-4D)sinx + (4C+3D)cosx]+ e^{2x}[(2C-D)sinx+(C+2D)cosx]
-6 e^{2x}(Csinx + Dcosx) = e^{2x} cosx
e^{2x} [(-C -5D)sinx + (5C – D)cosx] = e^{2x} cosx
C+ 5D = 0 ; 5C – D = 1⇒ C = 5/26 ; D = -1/26
y_p = e^{2x}(5/26 sinx -1/26 cosx)
y = $Ae^{-3x} + Be^{2x}$ + e^{2x}(5/26 sinx -1/26 cosx)

j. $\frac{d^2y}{dx^2}$ - 4 $\frac{dy}{dx}$ +4y = 8(x^2 + sin2x)

Auxiliary Equation: $r^2 - 4r + 4 = 0 \Rightarrow$ r = 2 ; r = 2

y_c = $(A + Bx)e^{2x}$

y_p = C + Dt + Et^2 + Fsin2t + Gcos2t;differentiating and substituting in the equation: C=3,D=4,E=2,F=0,G=1

y= $(A + Bx)e^{2x}$ + 2x^2+4x+3 + cos2x

k. $\frac{d^2y}{dx^2}$ + 4y = cos2x + cos4x

Auxiliary Equation: $r^2 + 4 = 0 \Rightarrow$ r = $\pm 2i$

y_c = $e^{\frac{-a}{2}x}(A\cos\alpha\, x + B\sin\alpha\, x)$; $\alpha = \frac{\sqrt{4b - a^2}}{2}$ = $\frac{\sqrt{4(4) - 0^2}}{2}$ = 2

y_c = $e^{\frac{-0}{2}x}(A\cos 2x + B\sin 2x)$; y_c = $A\cos 2x + B\sin 2x$

y_p = **x**(Csin2x + Dcos2x)+ Esin4x + Fcos4x **since sin2x is part of y_c** then differentiate and substitute in the equation:

y = $A\cos 2x + B\sin 2x$ + $\frac{1}{4}x\sin 2x$-$\frac{1}{12}\cos 4x$