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Matrices   Handout #8

Topic	Interpretation
Matrix Definition A matrix is an array of numbers: **A** = $\begin{pmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & ...... & a_{mn} \end{pmatrix}$ Matrices are denoted by capital letters : **A,B,C,…..** Matrix size or rank is determined by the number of rows × the number of columns it has. We say **A** has **m** rows and **n** columns or it is an **m×n** matrix.	*Example1:* **A** = $\begin{pmatrix} 2 & 0 \\ -5 & 3 \end{pmatrix}$ is **2 × 2** **B** = $\begin{pmatrix} 3 & 0 & -1 \\ 6 & 8 & 2 \\ 1 & 0 & 7 \\ -5 & -1 & 4 \end{pmatrix}$ is **4 × 3** **C** = $\begin{pmatrix} 1 & 6 & 5 & -2 & 3 \end{pmatrix}$ is **1 × 5**
Square Matrix A matrix with the same number of rows as columns: 2×2 , 3×3, 4×4 are all square matrices. **Identity Matrix** Has **1** in each of the positions in the main diagonal and **0** elsewhere. **Note that** : **I** is a Square matrix.	*Example2:* **I** = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the **2 × 2** Identity matrix **I** = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the **3 × 3** Identity matrix
Matrix Addition If **A** and **B** are two matrices of the same size then we define **A+B** to be the matrix whose elements are the sums of the corresponding elements in **A** and **B**. **Only matrices of the same size can be added.** **A + (B + C) = (A+B)+C** **A – B = A + (-B)** **k(A+B) = kA + kB**	*Example3:* $\begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 7 \\ -9 & 1 & -6 \\ 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 4 & 1 \\ 1 & 5 & 4 \\ 4 & 5 & 1 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 9 & 0 \\ 1 & 8 & 11 \\ -5 & 6 & -5 \\ 4 & 4 & 10 \end{pmatrix}$
Matrix Multiplication For the product of two matrices **A** and **B** to be defined, the number of columns of **A** must be the same as the number of rows in **B**:	*Example4:*

A : m×**n** ; B : **n**×p then **AB** is defined and of rank **m×p**

Properties

For any matrices **A , B , C** such that all the indicated sums and products exist:

A(BC) = (AB)C

A(B+C) = AB + BC

Remark

In general, AB may not equal BA.

Example5: suppose **A** is 2x**3** and **B** is **3**x5 then **AB** is defined, but **BA** is not defined. B(3x**5**) , A(**2**x3)

Remark

For any matrix **A** such that all the indicated products exist:

IA = AI = A

Where **I** is the identity matrix.

Determinant of a square matrix

To every **square** matrix **A**, there is an assigned number called the determinant of **A**. Written **det A** or **|A|**.

Determinant of a 2x2 matrix

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

|A| = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ ad – bc

Determinant of a 3x3 matrix

A= $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **then**

|A|= a $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ **- b** $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ **+c** $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Inverse Matrix

A **square** matrix **A** has an inverse $\mathbf{A^{-1}}$ if $\mathbf{AA^{-1} = A^{-1}A = I}$

Remarks

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 3 & 5 & 1 \\ 6 & 7 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 20 & 26 & 17 \\ 38 & 49 & 41 & 24 \end{pmatrix}$$

The product is obtained by multiplying each row of **A** by the columns of **B**(first by first and so on)

The first entry:

$(2 \quad 3 \quad 1)\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ = 2x1+ 3x2 +1x6 = 14

The second entry:

$(2 \quad 3 \quad 1)\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ = 2x2+ 3x3 +1x7 = 20

and so on ……

Example6:

1. **A=** $\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$; **|A|=** $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ =10-(-3)=13

2. **A=** $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$

|A| = 2 $\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix}$ **- 3** $\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix}$ **+(-4)** $\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}$

|A| = 2(-20 + 2) – 3(0 – 2) -4(0 +4)

|A| = 2(-18) -3(-2) -4(4)

|A| = -36 + 6 – 16 = - 46

Example7:

1.Only square matrices may admit an inverse.
2.When a square matrix has an inverse,it has only one(unique)
3.A square matrix may have **no** inverse.**If |A| = 0 then A^{-1} does not exist.**

Inverse of a 2x2 matrix

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a 3x3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We define the *cofactor* of a_{ij} denoted by A_{ij} as :
$A_{ij} = (-1)^{i+j}\,|M_{ij}|$
We call the determinant $|M_{ij}|$, the *minor* of a_{ij} .
The above matrix has 9 cofactors :

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

and so on ……

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Note the way in which A_{ij}'s are placed.

Solving Systems with matrices
Consider the system:
ax + by = c

Note that ,for a 2x2 matrix, the inverse is obtained by switching the “diagonal” terms a and d, changing the sign of the “off diagonal” terms b and c and finally dividing by the determinant of the matrix: ad – bc
If ad – bc = 0 ,then A^{-1} does not exist.

A = $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$; ad –bc = (1)(4)–(3)(-2)=10

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{10} & \frac{2}{10} \\ \frac{-3}{10} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{-3}{10} & \frac{1}{10} \end{pmatrix}$$

The matrix **A** = $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ has no inverse
since **|A|** = ad – bc = (2)(3) – (1)(6) = 0

<u>*Example8*</u>: **A** = $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ we know
|A| = -46 from Example6 above.

$A_{11} = (-1)^{1+1}\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = -(-2) = +2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$

Similarly: **A_{21}= -11, A_{22}=14, A_{23}=5, A_{31}=-10, A_{32}=-4, A_{33}=-8 ; then A^{-1} =**

$$\frac{1}{-46}\begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{pmatrix} = \begin{pmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ \frac{-1}{23} & \frac{-7}{23} & \frac{2}{23} \\ \frac{-2}{23} & \frac{-5}{46} & \frac{4}{23} \end{pmatrix}$$

Multiplying each entry by -1/46 .

a'x + b'y = c'

Let **A** = $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$; **X** = $\begin{pmatrix} x \\ y \end{pmatrix}$

and **B**= $\begin{pmatrix} c \\ c' \end{pmatrix}$

Since

AX = $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ = $\begin{pmatrix} ax+by \\ a'x+b'y \end{pmatrix}$

the original system is equivalent to the single matrix system : **AX = B**

$\mathbf{A^{-1}\,AX = A^{-1}\,B}$ (multiplying both sides by $\mathbf{A^{-1}}$)

$\mathbf{IX = A^{-1}B}$ ($\mathbf{A^{-1}A = I}$)

$\mathbf{X = A^{-1}B}$ (**IX = X**)

Conclusion

If $\mathbf{AX = B \Leftrightarrow X = A^{-1}B}$

That is, **AX = B** has a solution if and only if $\mathbf{A^{-1}}$ exists

This implies the following:

A square matrix **A** is invertible(has an inverse) if and only if **AX = B** has a unique solution.

Example9: Solve the system :

x + 1.5y = 8

2x + 3y = 10

A= $\begin{pmatrix} 1 & 1.5 \\ 2 & 3 \end{pmatrix}$, since the determinant of **A:**

|A| = (1)(3) – (2)(1.5) = 0

then $\mathbf{A^{-1}}$ **does not exist** and hence the above system has **no solution**.

Example10:

Solve the system :

-x -2y + 2z = 9

2x + y –z = - 3

3x – 2y + z = -6

We have **AX = B** where

A = $\begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix}$; **X** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$; **B** = $\begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix}$

Then $\mathbf{X = A^{-1}B}$

$\mathbf{A^{-1}}$ = $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{5}{3} & \frac{7}{3} & -1 \\ \frac{7}{3} & \frac{8}{3} & -1 \end{pmatrix}$

X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ = $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{5}{3} & \frac{7}{3} & -1 \\ \frac{7}{3} & \frac{8}{3} & -1 \end{pmatrix}\begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix}$ = $\begin{pmatrix} 1 \\ 14 \\ 19 \end{pmatrix}$

Thus the solution is (1, 14, 19)

Example11: Solve the matrix system

X = D + AX

X – AX = D with **X = IX**

IX –AX = D

(I – A)X = D

$\mathbf{X = (I - A)^{-1}\,D}$

Solving A system of three equations with three unknowns

when they ask for Matrix method ,you **can not** use Algebra , substitution , manipulation etc…
For example :
x+y+z = 6 ---------(1)
2x - y + z = 3 ----- (2)
x + z = 4 ---------- (3)
From (1) : z = 6 - x - y
Substitute for z in (2) and (3) :
(2): 2x - y + 6 - x - y = 3 implies x - 2y = -3 ----(4)
(3): x + 6 - x - y = 4 implies y = 2
putting this in (4) : x - 2(2) = - 3 implies x = 1
Finally : z = 6 - x - y = 6 - 1 - 2 = 3
Hence (x,y,z) = (1,2,3)

Instead ,you need to construct the matrix (Read slowly and carefully, don't move to another step before understanding the previous one):

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

1 1 1 | 6 --------R1
2 -1 1| 3 --------R2
1 0 1 | 4 --------R3
You need to make it(Note the three zeros) :
1 ? ? | ?
0 1 ? | ?
0 0 ? | ?
The normal procedure is to manipulate this in **3 steps** :
R1 with R2 ; R1 with R3 ; R2 with R3
Our aim is to:
1.)Make the first element of the second row (which is 2) a zero (this can be done by playing with R1 and R2)
2.)Make the first element in the third row (which is 1) a zero(this can be done by playing with R1 and R3)
3.)Make the second element of the third row a zero we need to play with R2 and R3

Step1 : **R1 with R2** :
We do this by multiplying R1 by - 2 and then add it to R2 :
-2R1 + R2 : 0 - 3 - 1 | - 9
The Matrix becomes:
1 1 1 | 6 --------R1
0 -3 -1|- 9--------R2
1 0 1 | 4 --------R3
Note that we only replaced R2 ; R1 is left un tampered.
Step2 : **R1 with R3** :
to make the first element in the third row (which is 1) ,
it is enough to calculate :
-R1 + R3 : 0 -1 0 | -2
The matrix becomes :
1 1 1 | 6 --------R1
0 -3 -1 |- 9--------R2
0 -1 0 | -2 --------R3
Step3 : **R2 with R3**
to make the second element of the third row (which is - 1)
a zero
R2 - 3R3 : 0 0 -1 | - 3
The Matrix becomes :
1 1 1 | 6 --------R1
0 -3 -1 |- 9--------R2
0 0 -1 | -3 --------R3
Now the last row means :
0x + 0y - z = -3 implies z = 3
Second row :
0x - 3y - z = - 9 , substitute for z (this is called back-substitution):
-3y - 3 = - 9 implies y = 2
First row :
x + y + z = 6 implies x + 2 + 3 = 6 implies x = 1
Hence (x,y,z) = (1,2,3)

Special cases :

(1) Case of impossible solution : we say the system is ***inconsistent***

This occurs when the last row in the final matrix looks like:

0 0 0 | N where N is any number.

In this case 0 X z = N which is impossible

(2) Case of infinite number of solutions:

This occurs when the last row in the final matrix looks like:

0 0 0 | 0

In this case 0 X z = 0 , here z could be any real number , and the system has infinite number of solution
Assume z = t (t any real number)
e.g.

1 1 1 | 3
0 1 2 | 0
0 0 0 | 0

Here 0 X z = 0 , let z = t
Row2: y + 2z = 0 implies y = -2z = -2t
Row1: x +y + z = 3 implies x -2t + t = 3 , x = t +3
Hence (x,y,z) = (t+3,-2t,t) where t is any real number.