Sharjah Institute of Technology

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Curve Sketching Handout #5

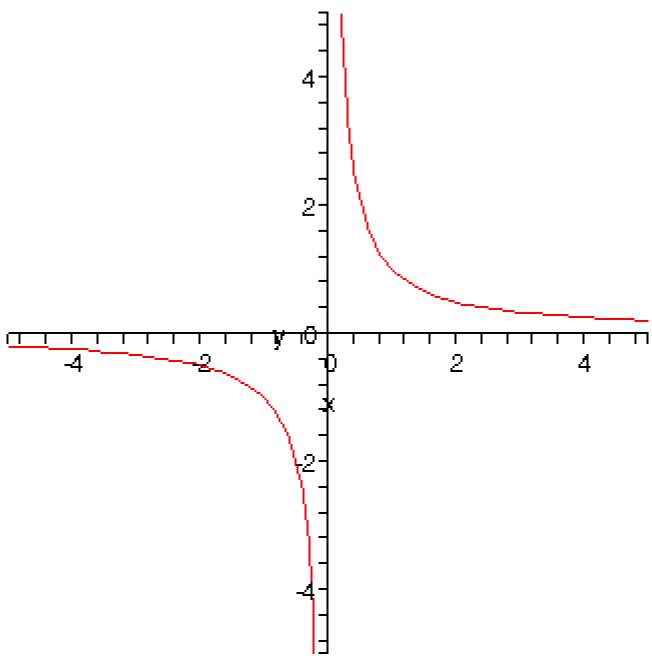
Topic
Interpretation

Rational Functions

A **rational function** is a function *f* that is a quotient of two polynomials. In other words, $f(x) = \frac{p(x)}{q(x)}$ is a rational function if $p(x)$ and $q(x)$ are polynomials and if $q(x)$ is not the zero polynomial.

The domain of $f(x)$ consists of all real numbers *x* for which $q(x) \neq 0$, since division by zero is undefined.

Consider the rational function $f(x) = \frac{1}{x}$. Its graph is below.



Notice how the graph looks like it has been ripped apart near $x = 0$. Not all graphs of rational functions separate like this. When they do we call the line of separation an **asymptote**.

Asymptotes

There are three major types of asymptotes.

- ***Vertical asymptotes*** which are also called **poles**
- ***Horizontal asymptotes***, and
- ***Oblique asymptotes***

We need to determine what asymptotes there are for a given rational function in order to graph it.

Vertical asymptotes of a rational function occur at any values of *x* that make the denominator equal to zero, provided that all common factors, other than constants have been eliminated.

Example 1: Find the vertical asymptotes of the following functions.

a. $f(x)=\dfrac{1}{x}$ There are no factors, other than 1, common to both the numerator and the denominator. So, set the denominator equal to zero and solve for *x*. So the vertical asymptote is the vertical line $x=0$.

b. $f(x)=\dfrac{x^2+x-6}{x-2}$. To find out if there are common factors, I need to completely factor the numerator and denominator. This produces the following. $f(x)=\dfrac{(x-2)(x+3)}{x-2}=x+3$. So, there is ***not*** a vertical asymptote for this rational function.

c. $f(x)=\dfrac{x}{x^2-4}$. Again, I need to completely factor both the numerator and denominator. This produces the following. $f(x)=\dfrac{x}{(x-2)(x+2)}$. So, the numerator and denominator have no common factors other than 1. Hence there are two vertical asymptotes and they are the vertical lines $x=\pm 2$.

Horizontal asymptotes of a rational function occur as follows.

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is always $y=0$.
- If the degree of the numerator and the denominator are the same, then the horizontal asymptote is determined by the ratio of the coefficients of the leading terms. For example, if $f(x)=\dfrac{ax^n+\cdots}{bx^n+\cdots}$, where ax^n and bx^n are the leading terms of the numerator and denominator respectively, then the horizontal asymptote occurs at $y=\dfrac{a}{b}$.
- If the degree of the numerator is greater than the degree of the denominator, no horizontal asymptote occurs.

Example 2: Find the horizontal asymptotes of the following functions.

a. $f(x)=\frac{1}{x}$. The degree of the numerator is 0 and the degree of the denominator is 1. Hence the horizontal asymptote is $y=0$.

b. $f(x)=\frac{x^2+x-6}{x-2}$. The degree of the numerator is 2 and the degree of the denominator is 1. Hence there is ***no*** horizontal asymptote.

c. $f(x)=\frac{3x^2}{x^2-4}$. The degree of the numerator and denominator is 2. Hence the horizontal asymptote is $y=\frac{3}{1}=3$

d. $f(x)=\frac{7x^3+9x^2-1}{2x^3+3x+10}$. The degree of the numerator and denominator is 3. Hence the horizontal asymptote is $y=\frac{7}{2}=3.5$.

An **oblique asymptote** occurs only when the degree of the numerator is greater than the degree of the denominator. If the difference of the degrees is only one, the oblique asymptote is a line (not vertical or horizontal). This is the only type of oblique asymptote we will discuss. However, please notice that a rational function cannot have both a horizontal and oblique asymptote.

The oblique asymptote is found by dividing the rational expression through the process of long division. The quotient obtained is the line which makes the oblique asymptote.

Example 3: Determine the oblique asymptotes of the following functions.

a. $f(x)=\frac{2x^2-3x-1}{x-2}$. Performing the long division we obtain the following.

$$\begin{array}{r} 2x+1 \\ x-2\overline{)2x^2-3x-1} \\ \underline{-(2x^2-4x)} \\ x-1 \\ \underline{-(x-2)} \\ 1 \end{array}$$

So the oblique asymptote is $y = 2x+1$

b. $f(x)=\dfrac{8x^3}{x^2+4}$ Performing the long division we obtain the following.

$$\begin{array}{r} 8x \\ x^2+0x+4\overline{)8x^3+0x^2+0x+0} \\ \underline{-(8x^3+0x^2+32x)} \\ -32x+0 \end{array}$$

So the oblique asymptote is $y=8x$.

c. $f(x)=\dfrac{x^3+2x^2-3x}{x^2-25}$ Performing the long division we obtain the following.

$$\begin{array}{r} x+2 \\ x^2+0x-25\overline{)x^3+2x^2-3x+0} \\ \underline{-(x^3+0x^2-25x)} \\ 2x^2+22x+0 \\ \underline{-(2x^2+0x-50)} \\ 22x+50 \end{array}$$

So the oblique asymptote is $y=x+2$

The Graph of a Rational Function

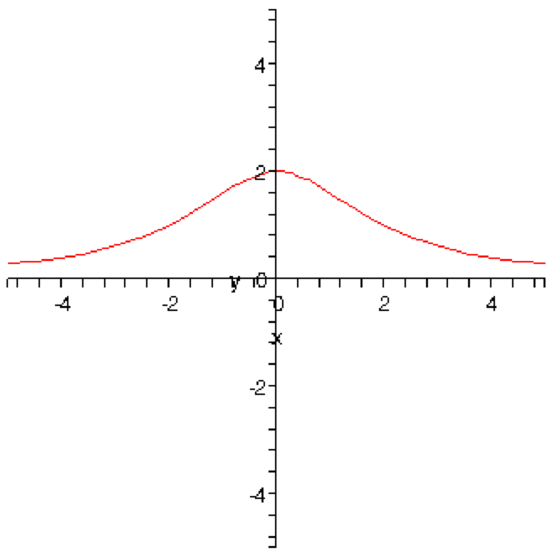
To graph a rational function $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ have no common factors:

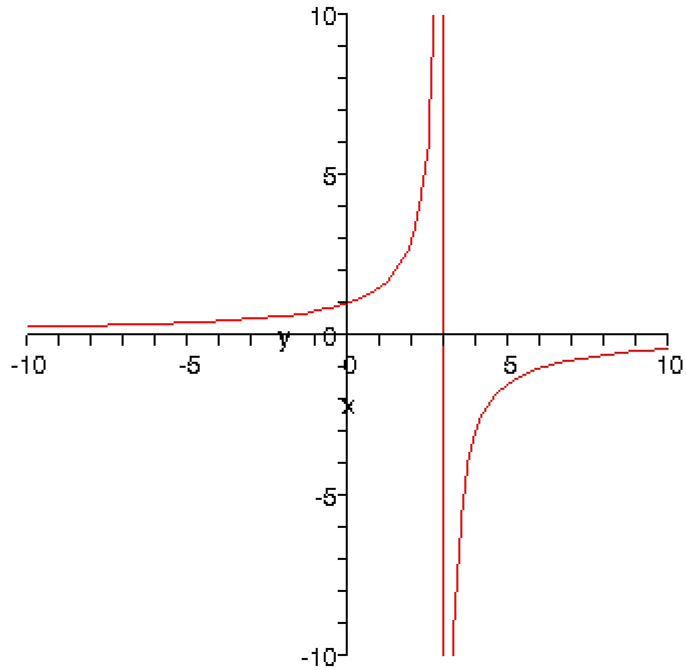
1. Find the vertical asymptotes and sketch them into the graph with a dotted line.
2. Find the horizontal or the oblique asymptote, if there is one, and sketch it into the graph with a dotted line.
3. Find and plot the zeros of the function. The zeros are the values of *x* that make the numerator equal to zero. In other words, the zeros of $f(x)$ are the zeros of $p(x)$.
4. Find and plot $f(0)$. This is the *y*-intercept of the function.
5. Find other function values to determine the general shape of the function. Then draw the graph.

Note: The graph of a rational function never crosses a vertical asymptote. However, it may or may not cross a horizontal or oblique asymptote. In addition, the end behavior of the graph will approach the horizontal or oblique asymptote.

Examples: Graph each of the following functions.

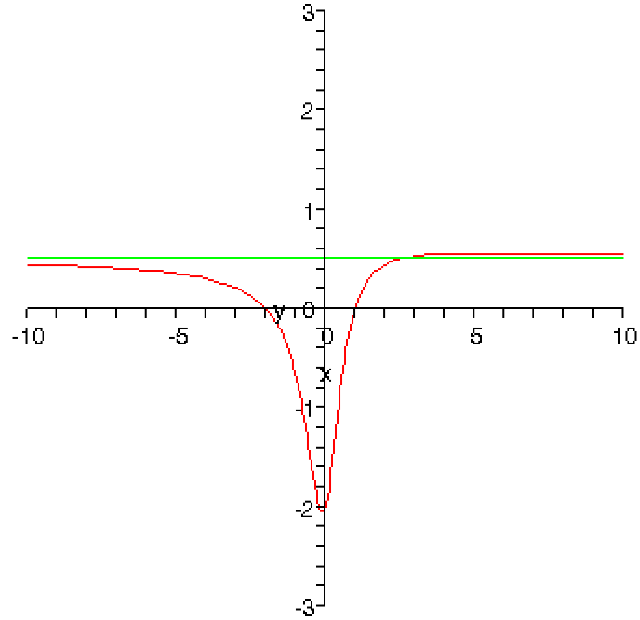
a. $f(x) = \dfrac{8}{x^2 + 4}$





b. $f(x) = \dfrac{3}{3 - x}$

c. $f(x)=\dfrac{x^2+x-2}{2x^2+1}$



d. $f(x)=\dfrac{x^3+2x^2-15x}{x^2-5x-14}$