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Sharjah Institute of Technology

Differential Equations2 Handout #16

Topic	Interpretation
Linear Equations $\frac{dy}{dt}$ + Py = Q Where P & Q are functions of t only. **Solution :** **y=** $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{dt}+c\right)$ <u>Example 2</u>: $\frac{dy}{dt}$ + 3y = 4 **y=** $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{dt}+c\right)$ **y=** $e^{\int -3dt}\left(\int 4e^{\int 3dt}\,\mathrm{dt}+c\right)$ y = e^{-3t} ($\int 4e^{3t}dt$ + c) y = e^{-3t} (4(1/3)e^{3t} + c) **Second order Equation:** $\frac{d^2y}{dt^2}$ + a$\frac{dy}{dt}$ + by = f(t) Similar situation to *Difference Equations.* The general solution : **y = y_c + y_p** where y_c is the complementary function and y_p is the particular integral. Auxiliary Equation: $r^2+ar+b=0$ **Case 1 :** r_1 and r_2 are real distinct. y_c = $Ae^{r_1t}+Be^{r_2t}$ **Case 2 :** r_1, r_2 are real and equal; r = r_1= r_2 yc = $(A+Bt)e^{rt}$ **Case 3 :** r_1, r_2 are imaginary y_c = $e^{\frac{-a}{2}t}(A\cos\alpha t+B\sin\alpha t)$	<u>Example1</u>: $\frac{dy}{dx}-2y=e^x$ P = -2 ; Q = e^x **y=** $e^{\int -Pdt}\left(\int Qe^{\int Pdt}\,\mathrm{dt}+c\right)$ **y=** $e^{\int 2xdx}\left(\int e^x e^{\int -2xdx}\,\mathrm{dx}+c\right)$ y = e^{2x}($\int e^x(\frac{-1}{2})e^{-2x}dx+c)$ y = e^{2x} (-½ $\int e^{-x}dx+c$) y = e^{2x} (½ e^{-x} + c) y = ½ e^x + ce^{2x} <u>Example 3</u>: $\frac{d^2y}{dt^2}$ - 3$\frac{dy}{dt}$ + 2y = te^{3t} Auxiliary Equation: $r^2-3r+2=0$ ⇒ r = 1 ; r = 2 y_c = Ae^t+Be^{2t} y_p = (C +Dt)e^{3t} to substitute this in the equation, we need y_p' and y_p'' y_p' =De^{3t} + 3(C+Dt)e^{3t} = (3C+D+3Dt)e^{3t} y_p'' = 3De^{3t} + 3(3C+D+3Dt)e^{3t} = (9C + 6D + 9Dt)e^{3t} Substituting in the equation: (9C + 6D+ 9Dt)e^{3t} -3(3C + D+ 3Dt)e^{3t} + 2(C +Dt)e^{3t} = te^{3t} (2C + 3D +Dt)e^{3t} = te^{3t} D = 1 ; 2C + 3D = 0 ; C = -3/2 General solution : **y = y_c + y_p** y = Ae^t+Be^{2t} + ($\frac{-3}{2}$ + t)e^{3t} <u>Example 4:</u>

where $\alpha = \dfrac{\sqrt{4b - a^2}}{2}$

PS : Because the solution depends on a and b; if the equation is given in the form :
2y'' + 6y' +4y= 0
You need to divide by 2 to get the correct values of a and b :
y'' + 3y'+ 2y = 0

Finding the Particular Solution :

f(t)	y_p
t^n e.g. t + 3 $2t^2$ 5	$A_0+A_1t+...+A_nt^n$ C + Dt C + Dt + Et^2 C+Dt
a^t e.g. 2^t 2^t + t	Ca^t $C2^t$ $C2^t$ + Dt + E
$a^t\ t^n$ e.g. 2^t (t)	$a^t(A_0+A_1t+....+A_nt^n)$ 2^t(C + Dt)
a^t sinbt e.g. sin2t	a^t(Acosbt + Bsinbt) Ccos2t + Dsin2t
a^t cosbt e.g. cosπ t	a^t(Acosbt + Bsinbt) Ccosπ t + Dsinπ t

Examples :
f(t) = 3t + 2 ; y_p= C + Dt
f(t) = t^3 ; y_p= C + Dt + Et^2 + Ft^3
f(t) = 2cos3t + t ;
y_p= Ccos3t + D sin3t + E + Ft;
f(t) = e^{2t} (3- 5t) ; y_p= e^{2t} (C+Dt)
f(t) = 3 + sin2t + e^{-3t}
y_p=C+Dt + Ecos2t+ Fcos2t+ Ge^{-3t}

f(t) = t^2sin 6t + (2t – 1)cos 6t
take the highest polynomial degree
$y_p=(C+Dt+Et^2)\sin 6t+ (F+Gt+Ht^2)\cos 6t$

About Particular solutions
In some cases you may substitute the particular solution in the equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

Auxiliary Equation: $r^2 + 6r + 9 = 0$
$\Rightarrow$ r = -3; r= -3 ; tow equal roots.
y_c = $(A + Bx)e^{-3x}$
There is no need for a particular integral as f(x)= 0 .

Example 5:

$$\frac{d^2y}{dx^2} + y = e^x \; ; \; y(0) = 1 \; ; \; y'(0) = 11/2$$

Auxiliary Equation: $r^2 + 1 = 0$; r = $\pm i$
Complex roots.

y_c = $e^{\frac{-a}{2}x}(A\cos\alpha\, x + B\sin\alpha\, x)$

$\alpha = \dfrac{\sqrt{4b - a^2}}{2} = \dfrac{\sqrt{4(1) - 0^2}}{2} = 1$

y_c = $e^{\frac{-0}{2}x}(A\cos x + B\sin x)$
y_c = $A\cos x + B\sin x$

$y_p = Ce^x$; $y_p' = Ce^x$; $y_p'' = Ce^x$

$$\frac{d^2y}{dx^2} + y = e^x \Rightarrow Ce^x + Ce^x = e^x$$

2 Ce^x = e^x $\Rightarrow$ C = ½ $\Rightarrow$ y_p = ½ e^x
General solution : **y = y_c + y_p**
y= $A\cos x + B\sin x$ + ½ e^x

y(0) = 1 $\Rightarrow$ Acos0 + Bsin0 + ½e^0 = 1

$\Rightarrow$ A + ½ = 1 $\Rightarrow$ A = ½

y'(0) =11/2 $\Rightarrow$-Asin0+ Bcos0+½e^0= 11/2
$\Rightarrow$ B+ ½ =11/2 $\Rightarrow$ B = 5

y= $\frac{1}{2}\cos x + 5\sin x$ + ½ e^x

Examples 6:

and get no values for the constants.This occurs when the **particular solution is part of the complementary function.**

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^{4x}$$

$r^2 - 5r + 4 = 0 \Rightarrow r = 1 \; ; \; r = 4$

$y_c = Ae^{x} + Be^{4x}$

Note: e^{4x} is part of y_c

The particular solution $y_p = Ce^{4x}$ will not work!

$y_p = Ce^{4x}$; $y_p' = 4Ce^{4x}$; $y_p'' = 16Ce^{4x}$

Substituting in the equation

$16Ce^{4x} - 20Ce^{4x} + 4Ce^{4x} = e^{4x}$

$\Rightarrow (0)e^{4x} = e^{4x}$??

To fix it we attach x to Ce^{4x}:

Let $y_p = Cxe^{4x}$; $y_p' = (4Cx+C)e^{4x}$;

$y_p'' = (16Cx+8C)e^{4x}$

$(16Cx+8C)e^{4x} - 5(4Cx+C)e^{4x} + 4Cxe^{4x} = e^{4x}$

$3Ce^{4x} = e^{4x} \Rightarrow C = 1/3$

$\Rightarrow y_p = (1/3)xe^{4x}$ **(see examples 6)**

Conditions for oscillation:

assume the roots of the auxiliary equation are r_1 and r_2;time path is oscillating if both roots are complex.

Conditions for convergence:

1. Two real distinct roots:

Both *negative* : $r_1 < 0$ and $r_2 < 0$ **Converges.**

If one of the roots is *positive*; **Diverges.**

2. Two real equal roots :

If the repeated root is *negative*; **Converges.**

If the repeated root is *positive*; **Diverges.**

3. Complex roots :

$e^{kt}(A\cos\alpha t + B\sin\alpha t)$

If $K < 0$; **converges.**

As t-->∞ , e^{-t} -----> 0

As t-->∞ , e^{t}-->∞ (see examples7)

1.) $y'' + 3y' - 10y = 3t + e^{-5t} - 1$

The auxiliary roots : $r = 2$, $r = -5$

$y_c = Ae^{2t} + Be^{-5t}$

The original particular solution:

$y_p = C + Dt + Ee^{-5t}$ **will not work! Since** e^{-5t} is part of the complimentary function.

To fix it we attach t to Ee^{-5t}: the correct one : $y_p = C + Dt + Ete^{-5t}$

2.) $y'' - 9y = 5t^2e^{3t} + t\cos t - \sin t$

$y_c = Ae^{-3t} + Be^{3t}$

The original particular solution:

$y_p = (C+Dt+Et^2)e^{3t} + (Ft+G)\cos t + (H+It)\sin t$ **will not work! Since** e^{3t} is part of the complimentary function.

To fix it we attach t to $(C + Dt + Et^2)e^{3t}$**:** the correct one :

$y_p = t(C+Dt+Et^2)e^{3t} + (Ft+G)\cos t + (H+It)\sin t$

3.) $y'' + 4y' + 4y = (2-3t^2)e^{-2t}$

$y_p = t^2(C+Dt+Et^2)e^{-2t}$ **why?**

<u>Examples 7:</u>1. $y = 5e^{-t} - 3e^{-2t} + 7$

Both roots are negative;converges.

As t----->∞ ; e^{-t} --> 0 ; e^{-2t} --> 0

y converges to 7 .

2. $y = -2e^{3t} - e^{-2t} + 2$

One of the roots : $3 > 0$;diverges.

As t--->∞ **; e^{3t} -->∞ ; e^{-2t} --> 0**

3. $y = e^{5t} - 3e^{2t} + t + 1$

Both roots are positive ;diverges.

As t--->∞ **; e^{5t} -->∞ ; e^{2t}-->∞**

4. $y = (2 + 3t)e^{4t}$;one positive repeated root ; diverges.

5. $y = (2 - t)e^{-7t}$ **;** one negative repeated root; converges.

PS: if both roots are negative or the repeated root is negative and there is a particular integral y_p then the behavior depends on y_p on the long run.

6. $y = 5e^{-t} - 3e^{-2t} + t + 1$

As t----->∞ **; e^{-t} --> 0 ; e^{-2t} --> 0**

On the long run it depends on t + 1.

7. $y = e^{-3t}(2\cos 5t + 4\sin 5t)$

$-3 < 0$; it converges.