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Taylor Series
Tutoring Sheet #6 – Solution

1. Obtain the expansion of the following functions as indicated :

a. f(x) = $e^{\frac{x}{2}}$ according to powers of x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \ldots\ldots.$$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \ldots\ldots\ldots$$

b. f(x) = lnx according to powers of x - 2 ;using Taylor's:

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots.$$

f(2) = ln2
f'(x) = x^{-1} $\Rightarrow$ f'(2) = ½
f''(x) = $-x^{-2}$ $\Rightarrow$ f''(2) = - ¼
f'''(x) = $2x^{-3}$ $\Rightarrow$ f'''(2) = ¼

$$\ln x = \ln 2 + \frac{x-2}{2} - \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{1}{4}\frac{(x-2)^3}{3!} - \ldots\ldots\ldots.$$

c. f(x) = $\cos^2 x$ according to powers of x

$$\cos^2 x = 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \ldots\ldots.$$

d. f(x) = $\frac{1}{1+x^2}$ according to powers of x

$\frac{1}{1+x^2}$ = 1 – x^2 + x^4 – x^6 + ……..

$\int_0^x \frac{dx}{1+x^2}$ = $\int_0^x (1-x^2+x^4-x^6+....)dx$ = x - $\frac{1}{3}x^3$ + $\frac{1}{5}x^5$ - $\frac{1}{7}x^7$

2. Use the expansions of e^{ix}, cosx and sinx to show that :

e^{ix} = cosx + isinx

e^x = 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + $\frac{x^5}{5!}$.....

$\Rightarrow e^{ix}$ = 1 + ix + $\frac{i^2x^2}{2!}$ + $\frac{i^3x^3}{3!}$ + $\frac{i^4x^4}{4!}$ + $\frac{i^5x^5}{5!}$ +…....

e^{ix} = 1 + ix - $\frac{x^2}{2!}$ - i$\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + i$\frac{x^5}{5!}$ +.....

= (1 - $\frac{x^2}{2!}$ + + $\frac{x^4}{4!}$ - $\frac{x^6}{6!}$ +....) + i(x - $\frac{x^3}{3!}$ + $\frac{x^5}{5!}$ +....)

= cosx + isinx

3. Evaluate using expansion, the following integral : $\int_0^1 \frac{\sin x}{x}$ dx

sinx = x - $\frac{x^3}{3!}$ + $\frac{x^5}{5!}$ - $\frac{x^7}{7!}$ ……….. ; $\frac{\sin x}{x}$ = 1 - $\frac{x^2}{3!}$ + $\frac{x^4}{5!}$ - $\frac{x^6}{7!}$ ………

$\int_0^1 \frac{\sin x}{x}$ dx = $\int_0^1 (1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^5}{7!}+...)$dx = x - $\frac{x^3}{3\times 3!}$ + $\frac{x^5}{5\times 5!}$ - $\frac{x^7}{7\times 7!}$ ... $\Big|_0^1$

= 0.946

4. Using the expansions of e^x and sinx , cosx ,find the expansions of the following:

a. $e^{1-\sin x} = e^{-1}e^{-\sin x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots; \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ldots$$

$-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} \ldots\ldots$ replace this by x in e^x :

$$e^{-\sin x} = 1 + (-x + \frac{x^3}{3!} \ldots) + \frac{1}{2!}(-x + \frac{x^3}{3!} \ldots)^2 + \frac{1}{3!}(-x + \frac{x^3}{3!} \ldots)^3$$

$$= 1 - x + \frac{x^3}{3!} \ldots\ldots + \frac{1}{2!}(x^2 - 2\frac{x^4}{3!} \ldots) + \frac{1}{3!}[(-x)^3 \ldots.]$$

$$= 1 - x + \frac{1}{2!}x^2 - \frac{x^4}{3!} \ldots\ldots.$$

$$e^{1-\sin x} = e^{-1}e^{-\sin x} = \frac{1}{e}(1 - x + \frac{1}{2!}x^2 - \frac{x^4}{3!} \ldots\ldots.)$$

b. $e^x \cos x$

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots..)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots..$$

$$+ x - \frac{x^3}{2!} + \frac{x^5}{4!} \ldots..$$

$$+ \frac{x^2}{2!} - \frac{x^4}{2!2!} + \frac{x^6}{2!4!} \ldots.$$

$$+ \frac{x^3}{3!} - \frac{x^5}{3!2!} + \frac{x^7}{3!4!}$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} \ldots\ldots\ldots$$