معهد الشارقة للتكنولوجيا
SHARJAH INSTITUTE OF TECHNOLOGY

Integration IV
Tutoring Sheet #18 – Solution

a. $\int x^3\left(x^2+1\right)^{3/2}dx$ **u = x² + 1 ⇒ du = 2xdx ⇒ dx =** $\frac{du}{2x}$

$\int x^3\left(x^2+1\right)^{3/2}dx = \int x^3 u^{3/2}\frac{du}{2x} = \frac{1}{2}\int x^2 u^{3/2}du$

But u = x² + 1 ⇒ x² = u – 1 , $\frac{1}{2}\int (u-1)u^{3/2}du$

$= \frac{1}{2}\int u^{5/2}du - \frac{1}{2}\int u^{3/2}du = \frac{1}{2}\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{1}{2}\frac{u^{5/2}}{\frac{5}{2}} + C$

$= \frac{1}{2}\times\frac{2}{7}u^{\frac{7}{2}} - \frac{1}{2}\times\frac{2}{5}u^{5/2} + C = \frac{1}{7}(x^2+1)^{7/2} - \frac{1}{5}(x^2+1)^{5/2} + C$

b. $\int x^5 e^{x^3}dx$, **u = x³ + 1 ⇒ du = 3x² dx ⇒ dx =** $\frac{du}{3x^2}$

$\int x^5 e^{x^3}dx = \int x^5 e^u \frac{du}{3x^2} = \frac{1}{3}\int x^3 e^u du$

But u = x³ , $\frac{1}{3}\int ue^u du$ **= ue^u – e^u + C (By Parts)**

= (1/3) $(x^3 e^{x^3} - e^{x^3}) + C$

c. $\int (x+2)^2 \ln x dx$, By parts

u = lnx ⇒ du = $\frac{1}{x}dx$

dv = (x+2)²

v = $\int (x+2)^2 dx = \int (x^2+4x+4)dx = \frac{x^3}{3} + 2x^2 + 4x$

$\int (x+2)^2 \ln x dx$, using $\int u dv = uv - \int v du$

$= (\frac{x^3}{3} + 2x^2 + 4x)\ln x$ - $\int \frac{\frac{x^3}{3} + 2x^2 + 4x}{x} dx$

Now : $\int \frac{\frac{x^3}{3} + 2x^2 + 4x}{x} dx = \int (\frac{x^3}{3} + 2x^2 + 4x)x^{-1} dx$

$= \int (\frac{x^2}{3} + 2x + 4) dx = \frac{1}{9}x^3 + x^2 + 4x + C$

$\int (x+2)^2 \ln x dx = (\frac{x^3}{3} + 2x^2 + 4x)\ln x - \frac{1}{9}x^3 - x^2 - 4x + C$

d. $\int \frac{dx}{1+e^x}$ **, u = 1+e^x $\Rightarrow$ du = e^x dx $\Rightarrow$ dx =** $\frac{du}{e^x}$

$= \int \frac{\frac{du}{e^x}}{u} = \int \frac{du}{ue^x}$ **, but u = 1+e^x $\Rightarrow$ e^x = u – 1**

$= \int \frac{du}{u(u-1)}$ **, By partial fractions**

$\frac{1}{u(u-1)} = \frac{a}{u} + \frac{b}{u-1}$

1 = a(u-1) + bu $\Rightarrow$ a = - 1 , b = 1

$\int \frac{du}{u(u-1)} = \int \frac{-du}{u} + \int \frac{du}{u-1}$ **= -ln|u| + ln|u-1| + C**

= -lne^x + ln|e^x-1| + C = -x + ln|e^x-1| + C

e. $\int \frac{\ln x dx}{x\left((\ln x)^2+1\right)}$ **, u = lnx $\Rightarrow$ du =** $\frac{1}{x}dx$

= $\int \frac{udu}{u^2+1}$ **, t = u^2 + 1 $\Rightarrow$ dt = 2udu**

= $\frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln\left|u^2+1\right| + C = \frac{1}{2}\ln((\ln x)^2+1) + C$

f. $\int \frac{x}{(x+1)^3}dx$ **, u = x+1 $\Rightarrow$ du = dx and x = u – 1**

= $\int \frac{u-1}{u^3}du = \int (u-1)u^{-3}du = \int (u^{-2}-u^{-3})du$

= $\frac{u^{-1}}{-1} - \frac{u^{-2}}{-2} + C = \frac{-1}{u} + \frac{1}{2u^2} + C = \frac{-1}{x+1} + \frac{1}{2(x+1)^2} + C$

g. $\int \sin^3 x \cos^5 x dx$

u = cosx $\Rightarrow$ du = - sinx dx $\Rightarrow$ dx = $\frac{-du}{\sin x}$

$\int \sin^3 x \cos^5 x dx = \int \sin^3 x u^5 \times \frac{-du}{\sin x} = -\int \sin^2 x u^5 du$

But $\sin^2$x = 1- $\cos^2$x = 1 – u^2

$-\int \sin^2 x u^5 du = -\int u^7 du = -\frac{u^8}{8} + C = \frac{-\cos^8 x}{8} + C$

h. $\int \frac{dx}{\sqrt{x}\left(x-5\sqrt{x}+6\right)}$ **, u =** $\sqrt{x} \Rightarrow$ **du =** $\frac{dx}{2\sqrt{x}}$

dx = 2$\sqrt{x}$ du

$\int \frac{dx}{\sqrt{x}\left(x-5\sqrt{x}+6\right)} = \int \frac{2\sqrt{x}du}{\sqrt{x}(x-5u+6)} = 2\int \frac{du}{x-5u+6}$

But u = $\sqrt{x} \Rightarrow$ **u^2 = x**

$2\int \frac{du}{x-5u+6} = 2\int \frac{du}{u^2-5u+6}$ **, By partial fractions**

$$\frac{1}{u^2-5u+6}=\frac{1}{(u-2)(u-3)}=\frac{a}{u-2}+\frac{b}{u-3}$$

1 = a(u-3) + b(u-2) $\Rightarrow$ a = -1 , b = 1

$$2\int\frac{du}{u^2-5u+6}=2\int\frac{-dx}{u-2}+2\int\frac{dx}{u-3}=-2\ln|u-2|+2\ln|u-3|+C$$

$$=2\ln|u-3|+2\ln|u-2|+C=2\ln\left|\frac{u-3}{u-2}\right|+C=\ln\left(\frac{u-3}{u-2}\right)^2+C=$$

$$\ln\left(\frac{\sqrt{x}-3}{\sqrt{x}-2}\right)^2+C$$

i. $\int\frac{dx}{\sqrt{\tan x}\cos^2 x}$ **, u = tanx $\Rightarrow$ du =** $\frac{dx}{\cos^2 x}$

$$=\int\frac{du}{\sqrt{u}}=\int u^{-1/2}du=\frac{u^{1/2}}{1/2}+C=2\sqrt{u}+C=2\sqrt{\tan x}+C$$

f. $\int\frac{\sqrt{1+\sqrt{e^{-x}}}}{\sqrt{e^x}}dx$ **, u = 1 +** $\sqrt{e^{-x}}$ **$\Rightarrow$ du =** $\frac{-e^{-x}}{2\sqrt{e^{-x}}}dx$

$\Rightarrow$ du = $\frac{-e^{-x}}{2\sqrt{e^{-x}}}dx=-\frac{dx}{2\sqrt{e^x}}$

Since $\left(\frac{-e^{-x}}{2\sqrt{e^{-x}}}=\frac{-\sqrt{(e^{-x})^2}}{2\sqrt{e^{-x}}}=\frac{1}{2}\sqrt{\frac{(e^{-x})^2}{e^{-x}}}=\frac{1}{2}\sqrt{e^{-x}}=\frac{1}{2\sqrt{e^x}}\right)$

$$\int\frac{\sqrt{1+\sqrt{e^{-x}}}}{\sqrt{e^x}}dx=-2\int\sqrt{u}du=-2\int u^{1/2}du=-2\frac{u^{3/2}}{3/2}+C$$

$$=-\frac{4u^{3/2}}{3}+C=-\frac{4\left(1+\sqrt{e^{-x}}\right)^{3/2}}{3}+C$$