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# Maxima&Minima
# Tutoring Sheet #14 – Solution

**1.** Find the maxima and the minima of the following functions:

a. $x^2$

$f'(x) = 2x = 0 \Rightarrow x = 0$

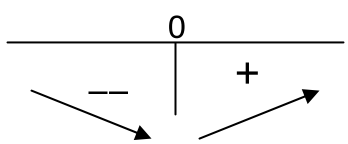
$f''(x) = 2 \Rightarrow f''(0) = 2 > 0 \Rightarrow x = 0$ minimizes f(x)

b. $2x^4 + 4$

$f'(x) = 8x^3 = 0 \Rightarrow x = 0$

$f''(x) = 24x^2 \Rightarrow f''(0) = 0 \Rightarrow$ no decision.

We have to study the sign of $f'(x) = 8x^3$




x = 0 minimizes f(x)

c. $x^3 – x$

$$f'(x) = 2x^2 – 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

$$f''(x) = 4x \Rightarrow f''\left(\frac{-1}{\sqrt{2}}\right) = \frac{-4}{\sqrt{2}} < 0$$

$$\Rightarrow x = \frac{-1}{\sqrt{2}} \text{ minimizes } f(x)\ ;\ f''\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} > 0$$

$$\Rightarrow x = \frac{1}{\sqrt{2}} \text{ maximizes } f(x).$$

d. $x^2 + 2x + 1$

$f'(x) = 2x+2 = 0 \Rightarrow x = -1$

$f''(x) = 2 \Rightarrow f''(-1) = 2 > 0 \Rightarrow x = -1$ minimizes f(x)

e. $2 + 4x - x^2$

$f'(x) = 4 - 2x = 0 \Rightarrow x = 2$

$f''(x) = -2 \Rightarrow f''(2) = -2 < 0 \Rightarrow x = 2$ maximizes f(x)

f. $2x^3 - 15x^2 + 36x + 4$

$f'(x) = 6x^2 - 30x + 36 = 0 \Rightarrow (6x - 12)(x - 3) = 0$

$\Rightarrow x = 2$ or $x = 3$

$f''(x) = 12x - 30 \Rightarrow f''(2) = -6 < 0 \Rightarrow x = 2$ maximizes f(x)

$f''(3) = 6 > 0 \Rightarrow x = 3$ minimises f(x).

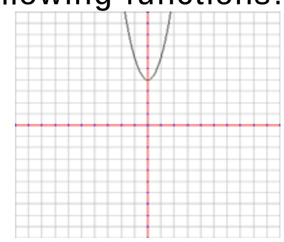
**2.** Find the maxima and the minima of the following functions:

a. $2x^2 + 4$

$f'(x) = 4x = 0 \Rightarrow x = 0$

$f''(x) = 4 \Rightarrow f''(0) = 4 > 0$

$\Rightarrow x = 0$ minimizes f(x)

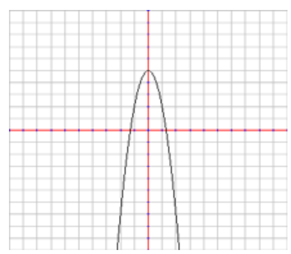


b. $5 - 3x^2$

$f'(x) = -6x = 0 \Rightarrow x = 0$

$f''(x) = -6 \Rightarrow f''(0) = -6 < 0$

$\Rightarrow x = 0$ maximizes f(x)



c. $2x^3 - 9x^2 - 24x + 10$

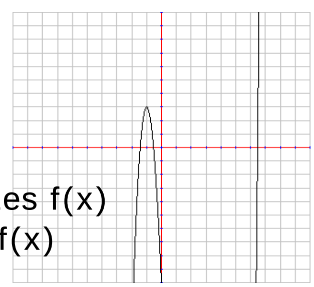
$f'(x) = 6x^2 - 18x - 24 = 0 \Rightarrow x = -1$

or $x = 4$

$f''(x) = 12x - 18$

$\Rightarrow f''(-1) = -30 < 0 \Rightarrow x = 0$ maximizes f(x)

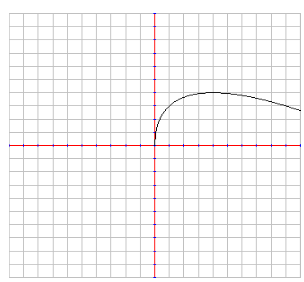
$f'(4) = 30 > 0 \Rightarrow x = 0$ minimizes f(x)



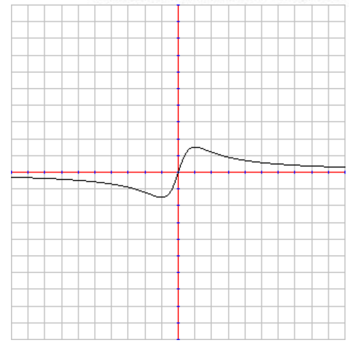
d. $4\sqrt{x} - x$

$f'(x) = \frac{4}{2\sqrt{x}} - 1 = 0 \Rightarrow x = 4$

$f''(x) = \frac{-1}{x\sqrt{x}}$ , $f''(x) = \frac{-1}{8} < 0$ (Max)



e. $\dfrac{3x}{x^2+1}$



f'(x) = $\dfrac{3(x^2+1)-(2x)(3x)}{(x^2+1)^2}$

f'(x) = $\dfrac{-3x^2+3}{(x^2+1)^2}$ = 0 $\Rightarrow$ $-3x^2 + 3 = 0$ $\Rightarrow$ x = - 1 or x = 1

f"(x) = $\dfrac{(-6x)(x^2+1)^2-(-3x^2+3)(2x)(x^2+1)}{(x^2+1)^4}$

f"(-1) = $\dfrac{6(4)-0}{16} > 0$ $\Rightarrow$ -1 minimizes f(x)

f"(1) = $\dfrac{-6(4)-0}{16} < 0$ $\Rightarrow$ 1 maximizes f(x)

**3.** Find all the local maxima and minima of the following functions,state whether each point is a maximum or minimum and find the value of the function at each point:

a. $y = x^2 - 4x + 2$
$y' = 2x - 4 = 0 \Rightarrow x = 2$
$y'' = 2 > 0 \Rightarrow x = 2$ minimizes the function.
To get the value of this minimum , substitute x = 2 in y :
$y = 2^2 - 4(2) + 2 = -4$

b. $y = x^3 - 3x^2$
$y' = 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$
$y'' = 6x - 6$

For x = 0 , y" = -6 < 0 $\Rightarrow$ x = 0 maximizes the function.
value of this maximum : $y = 0^3 - 3(0^2) = 0$

For x = 2 , y" = 6 > 0 $\Rightarrow$ x = 2 minimizes the function.
value of this minimum : $y = 2^3 - 3(2^2) = -4$

c. $y = x + \frac{1}{x}$

$y' = 1 + \frac{-1}{x} = 1 - \frac{1}{x^2} \Rightarrow x^2 - 1 = 0 \Rightarrow x = -1$ or $x = 1$

$y'' = \frac{2}{x^3}$

For $x = -1$ , $y'' = -2 < 0 \Rightarrow x = -1$ maximizes the function.

value of this maximum : $y = -1 + \frac{1}{-1} = -2$

For $x = 1$ , $y'' = 2 > 0 \Rightarrow x = 1$ minimizes the function.

Value of this minimum : $y = 1 + \frac{1}{1} = 2$

d. $y = x^5$

$y' = 5x^4 = 0 \Rightarrow x = 0$

$y'' = 20x^3$

For $x = 0$ , $y'' = 0 \Rightarrow$ Second Derivative Test *Fails*.

We need to study the sign of the *First* derivative:

$y' = 5x^4 > 0 \ \forall x$ : + 0 +

$\Rightarrow$ No maximum or minimum at this inflexion point.

**4.** Find the maximum value of the following functions(show it's maximum):

a. a. $f(x) = (1+x)e^{\frac{-x}{2}}$ of the form u.v

$u = 1 + x \Rightarrow u' = 1$ ; $v = e^{\frac{-x}{2}} \Rightarrow v' = \frac{-1}{2}e^{\frac{-x}{2}}$

$f'(x) = u'v + v'u = (1)\ e^{\frac{-x}{2}} + \frac{-1}{2}e^{\frac{-x}{2}}(1+x)$

$f'(x) = e^{\frac{-x}{2}}\left(1-\frac{1+x}{2}\right) = e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right) = 0 \Rightarrow x = 1$

To verify it is a maximum ,use second derivative test:

$f'(x) = e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right) \Rightarrow f''(x) = \frac{-1}{2}e^{\frac{-x}{2}}\left(\frac{1-x}{2}\right) + \frac{-1}{2}e^{\frac{-x}{2}}$

$\Rightarrow$ f''(1) = 0 - $\frac{1}{2}e^{\frac{-1}{2}}$ < 0 $\Rightarrow$ x = 1 maximizes f(x) .

To find the maximum ,substitute x = 1 in f(x)

f(1) = (1+1) $e^{\frac{-1}{2}}$ = 2$e^{\frac{-1}{2}}$ = $\frac{2}{\sqrt{e}}$

b. f(x) = x - x ln x

f'(x) = 1 – [(1)(lnx) + (x)($\frac{1}{x}$)] = 1 – (lnx +1) = -lnx = 0

-lnx = 0 $\Rightarrow$ lnx = 0 $\Rightarrow$ x = $e^0$ = 1 **(Recall: lnx = a $\Rightarrow$ x = $e^a$)**

To verify it is a maximum ,use second derivative test:

f''(x) = - $\frac{1}{x}$ $\Rightarrow$ f''(1) = -1 < 0 $\Rightarrow$ x = 1 maximizes f(x)

To find the maximum ,substitute x = 1 in f(x)

f(1) = 1 – (1)(ln1) = 1 – 0 = 1

**5.** Find the minimum value of the following functions(show it's minimum) :

a. f(x) = $e^{\sqrt{x}} - 2\sqrt{x}$ ; $e^{\sqrt{x}}$ is of the form $e^U$;its derivative is

U' $e^U$ ; the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$

f'(x) = $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ - 2($\frac{1}{2\sqrt{x}}$) = $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ - $\frac{1}{\sqrt{x}}$

= $\frac{1}{\sqrt{x}}$( ½ $e^{\sqrt{x}}$ - 1 ) = 0 $\Rightarrow$ ½ $e^{\sqrt{x}}$ - 1 = 0

$e^{\sqrt{x}}$ = 2 $\Rightarrow$ $\sqrt{x}$ = ln2 **(Recall: $e^x$ = a $\Rightarrow$ x = lna)**

x = $(\ln 2)^2$

To verify it is a minimum ,use second derivative test:

f'(x) = $\frac{1}{\sqrt{x}}$( ½ $e^{\sqrt{x}}$ - 1 ) is of the form u.v

u = $\frac{1}{\sqrt{x}}$ = $x^{-½}$ $\Rightarrow$ u' = - ½ $x^{-3/2}$

v = ½ $e^{\sqrt{x}}$ - 1 $\Rightarrow$ v' = ½ $\left(\frac{1}{2\sqrt{x}} e^{\sqrt{x}}\right) = \frac{1}{4\sqrt{x}} e^{\sqrt{x}}$

f''(x) = u'v + v'u = $\frac{-1}{4}(e^{\sqrt{x}} - 1)\, x^{-3/2} + \left(\frac{1}{4\sqrt{x}} e^{\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right)$

f''(x) = $\frac{-1}{4x^{\frac{3}{2}}}(e^{\sqrt{x}} - 1) + \frac{e^{\sqrt{x}}}{4x} > 0$

$e^{\sqrt{x}} = e^{\sqrt{(\ln 2)^2}} = e^{\ln 2} = 2$ **(Recall: $e^{\ln a} = a$)**

The value of the minimum : f($\ln^2 2$) = 2 – 2ln2

b. f(x) = $x^2 - \ln(\sqrt{2}\, x)$

**f'(x) =** $2x - \frac{\sqrt{2}}{\sqrt{2}x} = 2x - \frac{1}{x}$

f'(x) = 0 => $2x - \frac{1}{x} = 0$ => 2x = $\frac{1}{x}$ => $2x^2 = 1$

$x^2 = \frac{1}{2}$ => x = $\pm\frac{1}{\sqrt{2}}$

f''(x) = 2 + $\frac{1}{x^2} > 0\ \forall x$

x = $\pm\frac{1}{\sqrt{2}}$ both minimize f

F (x) = ln (cx)

=> F'(x) = $\frac{\mathbf{1}}{\boldsymbol{cx}} \times \boldsymbol{c} = \frac{\mathbf{1}}{\boldsymbol{x}}$

Example: f(x) = ln 3x

=> f'(x) = $\frac{\mathbf{1}}{\boldsymbol{x}}$