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Binomial Theorem Handout #7

1. EXPANSION OF BINOMIALS

The terms in any binomial expansion can be formed by tedious multiplication of terms, for example

$$(x + y)^2 = (x + y)(x + y).$$

If this is done, the following table results:

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

2. PASCAL'S TRIANGLE

Leaving out the x and y terms, the coefficients appear as follows:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5
1

The scheme for forming the coefficients is very simple. The first and last coefficients are always 1. To get any of the inner numbers, merely add the numbers on either side of it in the line above. For example, the 2 in line three is formed by adding the 1 on the left and the 1 on the right in the line above. The 3 in the fourth line is formed from the 1 and the 2 above it. Here are some examples:

$$(2a - 3b)^3 = [2a + (-3b)]^3 = (1)(2a)^3 + (3)(2a)^2(-3b)$$
$$+ (3)(2a)(-3b)^2 + (1)(-3b)^3$$
$$= 8a^3 - 36a^2b + 54ab^2 - 27b^3$$

$(pq - 4)^4 = (pq)^4 + 4(pq)^3(-4) + 6(pq)^2(-4)^2 + 4(pq)(-4)^3 + (-4)^4$

$= p^4q^4 - 16\ p^3q^3 + 96p^2q^2 - 256pq + 256$

3. FORMING BINOMIAL COEFFICIENTS

The formula for forming the r th $_nC_r = \dfrac{n!}{r!(n-r)!}$ binomial coefficient in the expansion to the n th power i

The binomial expansion can be written

$$(x+y)^n = {}_nC_0\ x^n + {}_nC_1\ x^{n-1}y + {}_nC_2\ x^{n-2}y^2 + \ldots + {}_nC_{n-1}\ xy^{n-1} + {}_nC_n\ y^n$$

For example, the second (remember, we count from 0) coefficient in an expansion of n = 5 is

$$_5C_2 = \frac{5!}{2!(5-2)!} = 10$$

Here, then, is how the expansion can look:

$$(x+y)^4 = {}_4C_0\ x^4 + {}_4C_1\ x^3y + {}_4C_2\ x^2y^2 + {}_4C_3\ xy^3 + {}_4C_4\ y^4$$

$$= x^4 + 4\ x^3y + 6\ x^2y^2 + 4\ xy^3 + y^4$$

4. AN EXAMPLE DRAWN FROM CALCULUS

If $f(x) = x^n$, find

$$\frac{f(x+h) - f(x)}{h} = \frac{x^n + nx^{n-1}h + \dfrac{n(n-1)x^{n-2}h^2}{2!} + \ldots - x^n}{h} = nx^{n-1} + \frac{n(n-1)x^{n-2}h^2}{2!} + \ldots$$

Note that when h approaches 0, this becomes nx^{n-1} .

5. A SHORT CUT

Multiply each term by the exponent of the preceding x-value and divide by the factorial of the term number.

$$(x+y)^n = \frac{x^n}{0!} + \frac{nx^{n-1}y}{1!} + \frac{n(n-1)x^{n-2}y^2}{2!} + \ldots$$

Here's still another way to write it:

$$(x+y)^4 = x^4 + 4x^3y + \frac{4 \bullet 3x^2y^2}{2} + \frac{4 \bullet 3 \bullet 2xy^2}{6} + \frac{4 \bullet 3 \bullet 2y^4}{24}$$

7. AN EXAMPLE FROM PHYSICS USING BINOMIAL EXPANSION

The velocity of a particle hanging from a spring which is pulled down a distance y from equilibrium and released is proportional to

$$\left(1+\frac{k}{y}\right)^{\frac{1}{2}}$$

where k is a constant depending on the mass of the particle, the spring characteristics, and g (the acceleration due to gravity). Expanding this expression using the binomial theorem gives

$$1+\frac{k}{2y}-\left(\frac{k}{2y}\right)^2+\ldots$$

Which can be simplified to

$$1+\frac{k}{2y}$$

since all terms after the second approach 0 very rapidly. Simplifications like this are among the most valuable uses of the binomial distribution.