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Taylor Series Handout #6

Taylor's Expansion

The Taylor polynomial for the function f(x) about x=a is

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots.$$

Maclaurin's Expansion

With a = 0 , $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots.$

Example: Expand f(x) = Arctanx = $\tan^{-1}x$

f(0) = Arctan 0 = 0 ; f'(x) = $\frac{1}{1+x^2} \Rightarrow$ f'(0) = 1 ; f''(x) = $\frac{-2x}{(1+x^2)^2} \Rightarrow$ f''(0)= 0 f'''(x) = -2 ,substituting all these in the Maclaurin's formula:

Arctanx = 0 + x(1) + $\frac{x^2}{2!}$(0) + $\frac{x^3}{3!}$(-2) …. = x - $\frac{x^3}{3}$ + $\frac{x^5}{5}$ - $\frac{x^7}{7}$ + …………

Famous Expansions :

e^x = 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ + ……….; sinx = x - $\frac{x^3}{3!}$ + $\frac{x^5}{5!}$ - $\frac{x^7}{7!}$ ……….

cosx = 1 - $\frac{x^2}{2!}$ + $\frac{x^4}{4!}$ - $\frac{x^6}{6!}$ ……….; ln(a+x) = lna + $\frac{x}{a}$ - $\frac{x^2}{2a^2}$ + $\frac{x^3}{3a^3}$ -………….

Note that expansion of lnx is not possible by Maclaurin's since the derivatives of lnx at x = 0 ,do not exist : f'(x) = 1 / x then f'(0) = 1 / 0?? However, the expansion of lnx about x = a (a$\neq$0) using **Taylor's** is possible :

lnx = lna + $\frac{1}{a}$(x-a) - $\frac{1}{2a^2}$(x-a)^2 + $\frac{1}{3a^3}$(x-a)^3 - ……; **e.g. lnx about x = 1**

ln x= ln1 + $\frac{1}{1}$(x-1) - $\frac{1}{2}$(x-1)^2 + $\frac{1}{3}$(x-1)^3 - ….=(x-1)- $\frac{1}{2}$(x-1)^2+ $\frac{1}{3}$(x-1)^3 - ….

Deducing Expansions Suppose we need the expansion of e^{-x} or e^{2x} or e^{-x^2} ,we can do this using the expansion of e^x without doing any computation : we have : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to get the expansion of e^{-x} simply replace x by –x in the expansion of e^x :

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

Example: Find the expansion of $e^{\cos x - 1}$ up to the term $\mathbf{x^4}$,deduce the expansion of $e^{\cos x}$;we have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots \Rightarrow \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ Now replace the whole expansion of (cosx–1) by x in the expansion of e^x :

$$e^{\cos x - 1} = 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \frac{1}{2!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^2 + \frac{1}{4!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^4$$

Note : for the square : find the first two terms only as in $(a-b)^2 = a^2 - 2ab$
for the Cube and up : cube only the first term .

$$e^{\cos x - 1} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{1}{2!}\left(\frac{x^4}{(2!)^2} - 2\frac{x^6}{2!4!} \ldots\right) + \frac{1}{4!}\left(\frac{x^8}{(2!)^4} \ldots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{1}{2!}\left(\frac{x^4}{(2!)^2}\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{6} + \ldots \text{(only up to } x^4\text{)}$$

$$e^{\cos x} = e(e^{\cos x - 1}) = e\left(1 - \frac{x^2}{2!} + \frac{x^4}{6} + \ldots\right)$$

Example : find the expansion of $e^x \sin x$ up to $\mathbf{x^5}$

we have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$

$e^x \sin x$ **=** $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\right)$ **Multiply :**

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + x^2 - \frac{x^4}{3!} \ldots + \frac{x^3}{2!} - \frac{x^5}{2!3!} = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} \ldots$$