# Calculus Applications Handout #19

## Applications of Calculus

### (A) Applications of the derivative:

#### 1. Related Rates

If two variables both vary with respect to time and have a relation between them, we can express the rate of change of one in terms of the other.
Strategy : differentiate both sides w.r.t. time , t .

**Example1**

The tuning frequency f of an electronic tuner is inversely proportional to the square root of the capacitance C in the circuit.
If f = 920 kHz for C = 3.5 pF, find how fast f is changing at this frequency if dC/dt = 0.3 pF/s.

Substituting the given values,

$$f = \frac{k}{\sqrt{C}} \Rightarrow 920000 = \frac{k}{\sqrt{3.5\times 10^{-12}}} \Rightarrow$$ **k = 1.721**

$$f = \frac{1.721}{\sqrt{C}} = 1.721C^{-1/2}$$

We need to find df/dt.With C = 3.5 pF and dC/dt = 0.3 pF/s.

$$\frac{df}{dt} = \frac{-1}{2}\times 1.721C^{-3/2}\frac{dC}{dt} = \frac{-1.721}{2}(3.5\times 10^{-12})^{-3/2}(0.3\times 10^{-12})$$

$$= -3942.8 \text{ Hz/s}$$

Therefore the frequency of the electronic tuner is decreasing at the rate of 39.4 kHz $s^{-1}$.

## 2. Curvilinear Motion

The velocity from the displacement function using:

$$v = \frac{ds}{dt}$$

and the acceleration from the velocity function (or displacement function), using:

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

These formulae are only appropriate for rectilinear motion (i.e. velocity and acceleration in a straight line). This is inadequate for most real situations, so we introduce here the concept of curvilinear motion, where an object is moving in a plane along a specified curved path. We generally express the x and y components of the motion as functions of time. This form is called parametric form.

| Physical Quantity | Horizontal Component | Vertical Component | Magnitude of the Resultant | Direction $\theta$ |
|---|---|---|---|---|
| Velocity | $v_x = \frac{dx}{dt}$ | $v_y = \frac{dy}{dt}$ | $v = \sqrt{v_x^2 + v_y^2}$ | $\tan\theta_v = \frac{v_y}{v_x}$ |
| Acceleration | $a_x = \frac{dx}{dt}$ | $a_y = \frac{dy}{dt}$ | $a = \sqrt{a_x^2 + a_y^2}$ | $\tan\theta_v = \frac{a_y}{a_x}$ |

***Example2***

A particle moves along the path $y = x^2 + 4x + 2$ where units are in centimetres. If the horizontal velocity $\boldsymbol{v_x}$ is constant at 3 cm $s^{-1}$, find the magnitude and direction of the velocity of the particle at the point (-1, -1)

In this example we have y in terms of x, and there are no expressions given in terms of "t" at all.

To be able to find magnitude and direction of velocity, we will need to know

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt}$$

Given that . $v_x = \frac{dx}{dt} = 3$ we need $\frac{dy}{dt}$

To find this, we differentiate the given function with respect to t throughout using the techniques we learned back in implicit differentiation:

$y = x^2 + 4x + 2$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} + 4\frac{dx}{dt} + 0$$

and we want to know the velocity at x = -1, we substitute these two values and get:

$$\frac{dy}{dt} = 2(-1)(3) + 4(3) = 6$$

So now we have $v_y = 6$ cm $s^{-1}$.Hence the magnitude of the velocity is given by:

$$\begin{aligned} v &= \sqrt{(v_x)^2 + (v_y)^2} \\ &= \sqrt{3^2 + 6^2} \\ &= 6.7082 \end{aligned}$$

The direction of the velocity is given by:

$$\begin{aligned} \theta_v &= \arctan\left(\frac{v_y}{v_x}\right) \\ &= \arctan\left(\frac{6}{3}\right) \\ &= 63.432°. \end{aligned}$$

## (B) Applications of the integral

### 1. Center of mass - Centroid

The moment of a mass is a measure of its tendency to rotate about a point. Clearly, the greater the mass (and the greater the distance from the point), the greater will be the tendency to rotate.The moment is defined as:

**Moment = mass × distance from a point**

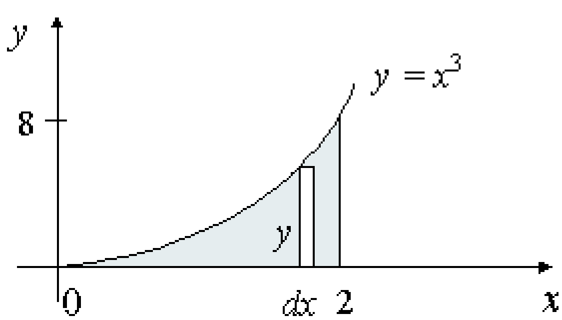
To find $\bar{x}$ (the x-coordinate of the centroid) and $\bar{y}$ (the y-coordinate) by taking moments about the y and x coordinates respectively.In general, we can say:

$$\bar{x} = \frac{\text{total moments } (x \text{ direction})}{\text{total area}}$$

$$\bar{y} = \frac{\text{total moments } (y \text{ direction})}{\text{total area}}$$

***Example3***

Find the centroid of the area bounded by $y = x^3$, $x = 2$ and the x-axis.



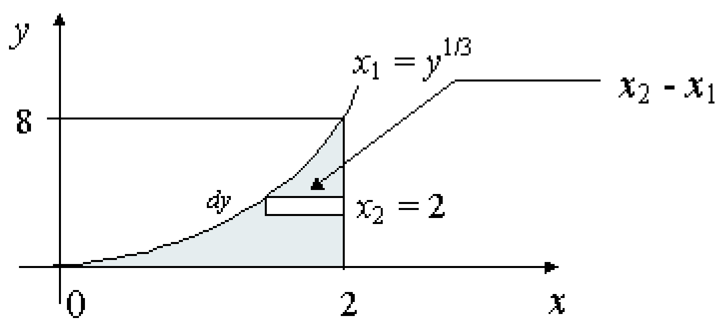

In this case, $y_2 = x^3$, $y_1 = 0$, $a = 0$, $b = 2$.

$$\bar{x} = \frac{\int_a^b x(y_2 - y_1)dx}{\int_a^b (y_2 - y_1)dx}$$

$$= \frac{\int_0^2 x(x^3 - 0)dx}{\int_0^2 (x^3 - 0)dx}$$

$$= \frac{\int_0^2 x^4 dx}{\int_0^2 x^3 dx} = \frac{\left[\frac{x^5}{5}\right]_0^2}{\left[\frac{x^4}{4}\right]_0^2} = \frac{32/5}{16/4} = 1.6$$

Now, for the y coordinate, we need to find:




$x_2$ = 2 (this is fixed in this problem)

$x_1 = y^{1/3}$ (this is variable in this problem)

c = 0, d = 8.

$$\bar{y} = \frac{\int_c^d y(x_2 - x_1)dx}{\int_c^d (x_2 - x_1)dx}$$

$$= \frac{\int_0^8 y(2 - y^{1/3})dy}{\int_0^8 (2 - y^{1/3})dy}$$

$$= \frac{\int_0^8 2y - y^{4/3})dy}{\int_0^8 (2 - y^{1/3})dy}$$

$$= \frac{\left[y^2 - \frac{3y^{7/3}}{7}\right]_0^8}{\left[2y - \frac{3y^{4/3}}{4}\right]_0^8} = 16 - \frac{3}{7}(32) = 2.29$$

The centroid is at (1.6, 2.29).

### 2. Work done by a variable force

The work (W) done by a constant force (F) acting on a body by moving it through a distance (d) is given by: $\boxed{\boldsymbol{W} = \boldsymbol{F} \times \boldsymbol{d}}$

If the force varies (e.g. compressing a spring) we need to use calculus to find the work done.If the force is given by F(x) (a function of x) then the work done by the force along the x-axis from a to b is:

$$W = \int_a^b F(x)dx$$

***Example4***

Find the work done on a spring when you compress it from its natural length of 1 m to a length of 0.75 m if the spring constant is k = 16 N/m.

**F = 16x :**

$$W = \int_0^{0.25} 16x\,dx$$

$$= \left[8x^2\right]_0^{0.25}$$

$$= 0.5 \text{ N.m}$$

**3. Electric Charges**

The force between charges is proportional to the product of their charges and inversely proportional to the square of the distance between them.

So we can write:

$$f(x) = \frac{kq_1q_2}{x^2}$$

where $q_1$ and $q_2$ are in coulombs, x is in meters, the force is in Newton and k is a constant, $k = 9 \times 10^9$.It follows that the work done when electric charges move toward each other (or when they are separated) is given by:

$$W = \int_a^b \frac{kq_1q_2}{x^2}\,dx$$

***Example5***

An electron has a $1.6 \times 10^{-19}$ C negative charge. How much work is done in separating two electrons from 1.0 pm to 4.0 pm?

(Recall: pm means picometre, or $10^{-12}$ metres)

$$W = \int_a^b \frac{kq_1q_2}{x^2}\,dx$$

$$= \int_{1\times10^{-12}}^{4\times10^{-12}} \frac{(9\times10^9)(-1.6\times10^{-19})^2}{x^2}\,dx$$

$$= (2.304\times10^{-28})\left[-\frac{1}{x}\right]_{1\times10^{-12}}^{4\times10^{-12}}$$

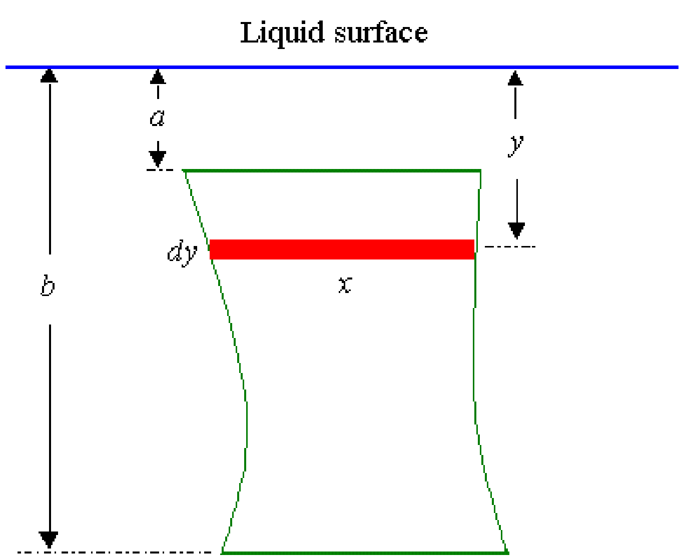
$$= 1.728\times10^{-16}\ \text{J}$$

**4. Force due to Liquid Pressure**

The force F on an area A at a depth y in a liquid of density w is given by

$$F = wyA$$

The force will increase if the density increases, or if the depth increases or if the area increases. So if we have an unevenly shaped plate submerged vertically in a liquid, the force on it will increase with depth. Also, if the shape of the plate changes as we go deeper, we have to allow for this. So we have:



Now, the total force on the plate is given by

$$F = w\int_a^b xy\,dy$$

where

x is the length (in m) of the element of area (expressed in terms of y)

y is the depth (in m) of the element of area

w is the density of the liquid (in N $m^{-3}$)

(for water, this is w = 9800 N $m^{-3}$)

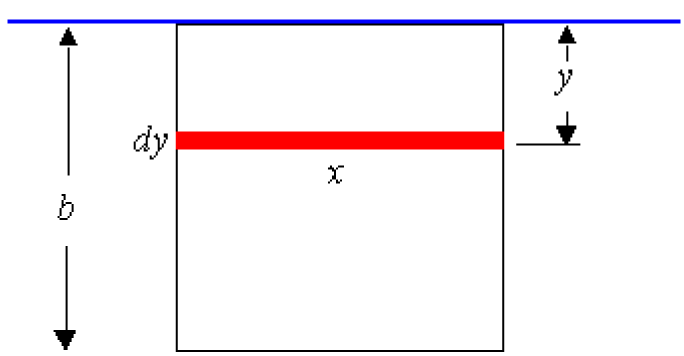
a is the depth at the top of the area in question (in m)

b is the depth at the bottom of the area in question (in m)

***Example6***

Find the force on one side of a cubical container 6.0 cm on an edge if the container is filled with mercury. The weight density of mercury is 133 kN/$m^3$.

This is the same as having a square plate of sides 6.0 cm submerged in mercury.



This is a very basic example where the width of the plate does not change as we move down the plate.

It is always x = 6.

Also, the depth of the top of the plate is 0, so a = 0.

To apply the formula, we have:

x = 0.06 m

y = is the variable of integration

w = 133 kN $m^{-3}$ = 133000 N $m^{-3}$

a = 0

b = 0.06

Therefore

$$\begin{aligned} F &= w\int_a^b xy\,dy \\ &= 133000\int_0^{0.06} 0.06y\,dy \\ &= 7980\int_0^{0.06} y\,dy \\ &= 7980\left[\frac{y^2}{2}\right]_0^{0.06} \\ &= 14.4 \text{ N} \end{aligned}$$