Handout #4

Basics IV: Logarithm/Exponential

Topic	Interpretation
The Exponential Function If a is a positive constant other than 1 ,then the function defined by : $f(x) = a^x$; $a > 0$; $a \neq 1$ is called exponential function with base a. **Properties:** 1. if $a^x = a^y \Leftrightarrow x = y$ 2. $a^x > 0$ for every x 3. All rules of indices apply; for example, $a^x \times a^y = a^{x+y}$ **Base e :** A special irrational number e **= 2.718281828**,arises naturally in many mathematical situations: $f(x) = e^x$ **Properties:** 1. $e^x > 0$ for every x; this means $\mathbf{e^x \neq 0}$. Also,$e^{-x} > 0$ (e raised to any power is positive) 2. $e^{-x} = \frac{1}{e^x}$ 3. All rules of indices apply; for example ,$(e^x)^2 = e^x \times e^x$ $= e^{x+x} = e^{2x}$ **and not** e^{x^2} 4. If $e^x = 1 \Rightarrow x = 0$ since $e^0 = 1$	e.g. $f(x) = 10^x$; $f(x) = 2^{-x}$; $f(x) = 3^{0.6x}$ If $a = 1 \Rightarrow f(x) = 1^x = 1$ Exponential functions with negative bases are not of interest because when a is negative,a^x may not be defined for some values of x ; e.g. $(-2)^{0.5} = \sqrt{-2}$ is not a real number. Example1: $2^{x+1} = 8 \Rightarrow 2^{x+1} = 2^3$ $\Leftrightarrow x + 1 = 3 \Rightarrow x = 2$ Example2 : $3^{-2} > 0$ since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ Example3: Solve for x : 1. $x^2 e^{\frac{-x}{2}} - e^{\frac{-x}{2}} = 0$; a good strategy is to isolate the exponential in any equation that involves them.i.e. take them as common factors : $(x^2 - 1) e^{\frac{-x}{2}} = 0$; since $e^{\frac{-x}{2}} \neq \mathbf{0}$ then $x^2 - 1 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$ 2. $e^x = -4$; No real solution since $e^x > 0$ Example4: 1. $(e^x + e^{-x})^2 = (e^x)^2 + 2(e^x)(e^{-x}) + (e^{-x})^2$ $= e^{2x} + 2e^{x-x} + e^{-2x}$ $= e^{2x} + 2e^0 + e^{-2x}$; with $e^0=1$; $= e^{2x} + e^{-2x} + 2$ 2. Solve $e^{2x} + 2e^x + 1 = 0$ $\Rightarrow (e^x - 1)^2 = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$

Logarithms

Any Base a : Exponential form

$$y = \log_a x \Leftrightarrow a^y = x$$

Logarithmic Form

y is the *logarithm* of x to the base a .

Note that x = a^y > 0 ;Hence In y = $\log_a x$, x > 0

i.e. y = $\log_a x$ is *defined* only for x > 0 .

Properties:

1. $\log_a 1 = 0$

   the logarithm of 1 to any base is zero.

2. $\log_a a = 1$

   the logarithm of the base is One.

3. $\log_a x^n = n \log_a x$

4. $\log_a x + \log_a y = \log_a xy$

5. $\log_a x - \log_a y = \log_a \dfrac{x}{y}$

6. $a^{\log_a x} = x$; e.g. $2^{\log_2 5} = 5$

Example:

A.)Write in the logarithmic form

1. $2^4 = 16 \Leftrightarrow \log_2 16 = 4$

base

Exponent base Exponent

2. $3^{-2} = 9 \Leftrightarrow \log_3 9 = -2$

3. $7^0 = 1 \Leftrightarrow \log_7 1 = 0$

B.)Write in the exponential form :

1. $\log_5 125 = 3 \Leftrightarrow 5^3 = 125$
2. $\log_3 1 = 0 \Leftrightarrow 3^0 = 1$

Example2: $\log_5(-2)$ does not exist.

$\log_2 0$ does not exist.

Quantity under log must be always > 0 .

1. Simply because $a^0 = 1 \Leftrightarrow \log_a 1 = 0$

   e.g. $\log_2 1 = 0$; $\log_6 1 = 0$ .etc….

2. Simply because $a^1 = a \Leftrightarrow \log_a a = 1$

   e.g. $\log_2 2 = 1$; $\log_5 5 = 1$

e.g. $\log_2 8 = \log_2 2^3 = 3\log_2 2 = 3(1) = 3$

e.g. $\log_8 3 + \log_8 5 = \log_8 15$

e.g. $\log_5 100 - \log_5 4 = \log_5 \dfrac{100}{4} = \log_5 25$

$= \log_5 5^2 = 2\log_5 5 = 2(1) = 2$

Base 10:

Written : $\log x$ i.e. without base attached to log means it is to base **10**.

$y = \log x \Leftrightarrow 10^y = x$

e.g. log 100 = log 10^2
= 2log10 = 2(1) = 2

e.g. $10^{\log 3}$ = 3

Base e : **y = Lnx ⇔ e^y = x**
Written : **ln x**

Change of base :Any base to base e :

$\log_a x = \frac{\ln x}{\ln a}$;

Some rules of log :

Base 10 : $\log x$	Base e : **ln x**
$\log 1 = 0$	**Ln 1 = 0**
$\log 10 = 1$	**Ln e = 1**
$\log x^n = n \log x$	**Ln x^n = n Ln x**
$\log x + \log y = \log xy$	**Ln x+Ln y= Lnxy**
$\log x - \log y = \log \frac{x}{y}$	**Lnx – Lny =** $\ln \frac{x}{y}$
$10^{\log x}$ = x	**$e^{\ln x}$ = x**

e.g. Ln e^3 = 3 Ln e = 3(1) = 3
e.g. $e^{\ln 5}$ = 5

e.g. $\log_7 x = \frac{\ln x}{\ln 7}$

Logarithmic/Exponential Equations

Recall :a^x > 0 ; e^x>0 (**e^x ≠ 0**)
Logx : x > 0
Most equations can be solved using the definitions:

1. $y = \log_a x \Leftrightarrow a^y = x$

e.g. $\log_2 x = 5 \Leftrightarrow x = 2^5 = 32$

e.g.
$\log_3(x^2 - 1) = 2 \Leftrightarrow x^2 - 1 = 3^2 = 27$
$\Rightarrow x^2 = 28 \Rightarrow x = \pm\sqrt{28} = \pm 2\sqrt{7}$

2. $\log x = a \Leftrightarrow x = 10^a$

e.g. log x = 3⇒x = 10^3 =1000

Examples:

1. $2e^{-x^2}$ -2xe^{-x^2} -4x2e^{-x^2} = 0
= $2e^{-x^2}$ (1 – x – $2x^2$)

⇒ -$2x^2$ – x + 1 = 0 ; a quadratic equation with
a = - 2 , b = -1 and c = 1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{9}}{-4} = \frac{1 \pm 3}{-4}$$

x = -1 or x = ½

2. e^{2x} +2 e^x – 3 = 0

$(e^x)^2$ + 2 e^x – 3 = 0 ; Let y = e^x > 0
y^2 + 2y – 3 = 0

e.g. log(x- 1) = 1

$\Rightarrow$ x – 1 = 10^1 = 10

$\Rightarrow$ x = 11

e.g. log (2x – 1) = log 5

$\Rightarrow$ 2x – 1 = 5

$\Rightarrow$ 2x = 6 $\Rightarrow$ x = 3

3. Lnx = a $\Leftrightarrow$ x = e^a

e.g. **Ln x = 2 $\Rightarrow$ x = e^2**

e.g. **Ln (x – 1) = 0**
$\Rightarrow$ x – 1 = e^0 = 1
$\Rightarrow$ x = 2

e.g. **Ln x – Ln (x+1) = 2**

$$\Rightarrow \ln\frac{x}{x+1} = 2\ln e = \ln e^2$$

(since aLn x = $\ln x^a$)

$$\Rightarrow \frac{x}{x+1} = e^2 \Rightarrow e^2 x + e^2 = x$$

$\Rightarrow$ e^2 x – x = - e^2

$\Rightarrow$ (e^2 – 1)x = -e^2

$$\Rightarrow x = \frac{-e^2}{e^2 - 1}$$

(y – 1) (y + 3) = 0
y = 1 or y = - 3
e^x = 1 $\Rightarrow$ x = Ln 1 = 0
e^x = -3 No solution since e^x > 0

3. $(\ln x)^2$ – lnx -2 = 0 ;Let y = lnx
y^2 – y – 2 = 0
(y + 1)(y - 2) = 0 ; y = -1 or y = 2
lnx = -1 $\Rightarrow$ x = e^{-1} = 1/e
lnx = -2 $\Rightarrow$ x = e^{-2} = 1/e^2

4. Ln x + Ln (x+1) = 1 ; with ln e = 1
Ln x(x+1) = ln e
x(x+1) = e $\Rightarrow$ x^2 + x – e = 0
a quadratic equation with
a = 1 , b = 1 and c = e

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4e^2}}{2}$$

No roots since 1 – 4e^2 < 0

5. 5e^x – e^{-x} = 4

$$5e^x - \frac{1}{e^x} = 4 \Rightarrow 5e^{2x} - 1 = 4e^x$$

$\Rightarrow$ 5e^{2x} - 4e^x – 1 = 0
a = 5 , b = -4 and c = -1

$$e^x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{36}}{10} = \frac{4 \pm 6}{10}$$

e^x = 1 $\Rightarrow$ x = ln1 = 0
e^x = -1/5 $\Rightarrow$no solution since e^x > 0

Caution: In general
Ln(x+y) $\neq$ Ln x + Ln y

Ln x – Ln y $\neq$ $\ln\frac{x}{y}$